

On stochastic calculus with respect to q -Brownian motion

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ABSTRACT. We pursue the investigations initiated by Donati-Martin [8] regarding stochastic calculus with respect to the q -Brownian motion, and essentially extend the previous results along two directions: (i) We develop a robust L^∞ -integration theory based on rough-paths principles and apply it to the study of q -Bm-driven differential equations; (ii) We provide a comprehensive description of the multiplication properties in the q -Wiener chaos.

Our presentation follows a probabilistic pattern, in the sense that it only leans on the law of the process and not on its particular construction. Besides, our formulation puts the stress on the rich combinatorics behind non-commutative processes, in the spirit of the machinery developed by Nica and Speicher in [14].

1. INTRODUCTION: THE q -BROWNIAN MOTION

The q -Gaussian processes (for $q \in [0, 1)$) stand for one of the most standard families of non-commutative random variables in the literature. Their consideration can be traced back to a paper by Frisch and Bourret in the early 1970s [9]: the dynamics is therein suggested as a model to quantify some possible commutation default between the creation and annihilator operators on the Fock space, the limit case $q = 1$ morally corresponding to the classical probability framework. The mathematical construction and basic stochastic properties of the q -Gaussian processes were then investigated in the 1990s, in a series of pathbreaking papers by Bożejko, Kümmerer and Speicher [3, 4, 5].

For the sake of clarity, let us briefly recall the framework of this analysis and introduce a few notations that will be used in the sequel (we refer the reader to the comprehensive survey [14] for more details on the subsequent definitions and assertions). First, recall that the processes under consideration consist in paths with values in a non-commutative probability space, that is a von Neumann algebra \mathcal{A} equipped with a weakly continuous, positive and faithful trace φ . The sole existence of such a trace φ on \mathcal{A} (to be compared with the “expectation” in this setting) is known to give the algebra a specific structure, with “ L^p ”-norms

$$\|X\|_{L^p(\varphi)} := \varphi(|X|^p)^{1/p} \quad (|X| := \sqrt{XX^*})$$

closely related to the operator norm $\|\cdot\|$:

$$\|X\|_{L^p(\varphi)} \leq \|X\| \quad , \quad \|X\| = \lim_{p \rightarrow \infty} \|X\|_{L^p(\varphi)} \quad , \quad \text{for all } X \in \mathcal{A} . \quad (1)$$

Now recall that non-commutative probability theory is built upon the following fundamental spectral result: any element X in the subset \mathcal{A}_* of self-adjoint operators in \mathcal{A} can be associated with a law that shares the same moments. To be more specific, there exists a unique compactly supported probability measure μ on \mathbb{R} such that for any real polynomial P ,

$$\int_{\mathbb{R}} P(x) d\mu(x) = \varphi(P(X)) . \quad (2)$$

Based on this property, elements in \mathcal{A}_* are usually referred to as (non-commutative) random variables, and in the same vein, the law of a given family $\{X^{(i)}\}_{i \in I}$ of random variables in (\mathcal{A}, φ) is defined as the set of all of its joint moments

$$\varphi(X^{(i_1)} \cdots X^{(i_r)}) \quad , \quad i_1, \dots, i_r \in I \quad , \quad r \in \mathbb{N} .$$

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With this stochastic approach in mind, the definition of a q -Gaussian family can be easily introduced along the following combinatorial description:

Definition 1.1. 1. Let r be an even integer. A pairing of $\{1, \dots, r\}$ is any partition of $\{1, \dots, r\}$ into $r/2$ disjoint subsets, each of cardinality 2. We denote by $\mathcal{P}_2(\{1, \dots, r\})$ or $\mathcal{P}_2(r)$ the set of all pairings of $\{1, \dots, r\}$.

2. When $\pi \in \mathcal{P}_2(\{1, \dots, r\})$, a crossing in π is any set of the form $\{\{x_1, y_1\}, \{x_2, y_2\}\}$ with $\{x_i, y_i\} \in \pi$ and $x_1 < x_2 < y_1 < y_2$. The number of such crossings is denoted by $\text{Cr}(\pi)$.

Definition 1.2. For any fixed $q \in [0, 1)$, we call a q -Gaussian family in a non-commutative probability space (\mathcal{A}, φ) any collection $\{X_i\}_{i \in I}$ of random variables in (\mathcal{A}, φ) such that, for every integer $r \geq 1$ and all $i_1, \dots, i_r \in I$, one has

$$\varphi(X_{i_1} \cdots X_{i_r}) = \sum_{\pi \in \mathcal{P}_2(\{1, \dots, r\})} q^{\text{Cr}(\pi)} \prod_{\{p, q\} \in \pi} \varphi(X_{i_p} X_{i_q}). \quad (3)$$

Therefore, just as with classical (commutative) Gaussian families, the law of a q -Gaussian family $\{X_i\}_{i \in I}$ is completely characterized by the set of its covariances $\varphi(X_i X_j)$, $i, j \in I$. In fact, when $q \rightarrow 1$ and φ is - at least morally - identified with the usual expectation, relation (3) is nothing but the classical Wick formula satisfied by the joint moments of Gaussian variables.

When $q = 0$, such a family of random variables is also called a semicircular family, in reference to its marginal distributions (see [14, Chapter 8] for more details on semicircular families, in connection with the so-called free central limit theorem).

We are now in a position to introduce the family of processes at the core of our study:

Definition 1.3. For any fixed $q \in [0, 1)$, we call q -Brownian motion (q -Bm) in some non-commutative probability space (\mathcal{A}, φ) any q -Gaussian family $\{X_t\}_{t \geq 0}$ in (\mathcal{A}, φ) with covariance function given by the formula

$$\varphi(X_s X_t) = s \wedge t. \quad (4)$$

In the same spirit as above, the q -Bm distribution can be regarded as a straightforward extension of two well-known processes:

- When $q \rightarrow 1$, one recovers the classical Brownian-motion dynamics, with independent, stationary and normally-distributed increments.
- The 0-Brownian motion coincides with the celebrated free Brownian motion, whose freely-independent increments are known to be closely related to the asymptotic behaviour of large random matrices, following Voiculescu's breakthrough results [17].

Thus, we have here at our disposal a family of processes which, as far as distribution is concerned, provides a natural “smooth” interpolation between two of the most central objects in probability theory: the standard and the free Brownian motions. It is then natural to wonder whether the classical stochastic properties satisfied by each of these two processes can be “lifted” on the level of this interpolation, or in other words if the properties known for $q = 0$ and $q \rightarrow 1$ can be extended to every $q \in [0, 1)$. Of course, any such extension potentially offers an additional piece of evidence in favor of this interpolation model, as a privileged link between the free and the commutative worlds.

Some first results in this direction, focusing on the stationarity property and the marginal-distribution issue, can be found in [3]:

Proposition 1.4. For any fixed $q \in [0, 1)$, let $\{X_t\}_{t \geq 0}$ be a q -Brownian motion in some non-commutative probability space (\mathcal{A}, φ) . Then for all $0 \leq s < t$, it holds that $X_t - X_s \sim \sqrt{t-s} X_1$. In particular, any q -Brownian motion $\{X_t\}_{t \geq 0}$ is a $\frac{1}{2}$ -Hölder path in \mathcal{A} , i.e.

$$\sup_{s < t} \frac{\|X_t - X_s\|}{|t - s|^{1/2}} \leq \|X_1\| < \infty. \quad (5)$$

Moreover, the law ν_q of X_1 is absolutely continuous with respect to the Lebesgue measure; its density is supported on $[\frac{-2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}]$ and is given, within this interval, by the formula

$$\nu_q(dx) = \frac{1}{\pi} \sqrt{1-q} \sin \theta \prod_{n=1}^{\infty} (1-q^n) |1 - q^n e^{2i\theta}|^2, \quad \text{where } x = \frac{2 \cos \theta}{\sqrt{1-q}} \text{ with } \theta \in [0, \pi].$$

A next natural step is to examine the possible extension, to all $q \in [0, 1)$, of the *stochastic integration* results associated with the free/classical Brownian motion. Let us here recall that the foundations of stochastic calculus with respect to the free Brownian motion (that is, for $q = 0$) have been laid in a remarkable paper by Biane and Speicher [2]. Among other results, the latter study involves the construction of a free Itô integral, as well as an analysis of the free Wiener chaoses generated by the multiple integrals of the free Brownian motion.

These lines of investigation have been followed by Donati-Martin in [8] to handle the general q -Bm case, with the construction of a q -Itô integral and a first study of the q -Wiener chaos. Let us also mention the results of [6] related to the extension of the fourth-moment phenomenon that prevails in Wiener chaoses.

Our aim in this paper is to go further with this analysis and extend the previous results in the two following directions:

(i) In the continuation of [7], we propose to adapt some of the main *rough-path principles* to this setting. The aim here is to derive a very robust integration theory allowing, in particular, to consider the study of differential equations driven by the q -Bm, i.e. sophisticated dynamics of the form

$$dY_t = f(Y_t) \cdot dX_t \cdot g(Y_t), \quad (6)$$

for smooth functions f, g . In fact, thanks to the general (non-commutative) rough-path results proved in [7] (and which we will recall in Section 2), the objective reduces to the exhibition of a so-called *product Lévy area* above the q -Bm, that is a kind of iterated integral of the process.

At this point, we would like to draw the reader's attention to the fact that the construction in [8] of a q -Itô integral as an element of $L^2(\varphi)$ is not sufficient for our purpose. Indeed, the rough-path techniques are based on Taylor-expansion procedures, which, for obvious stability reasons, forces us to consider an algebra norm in the computations. As a result, any satisfying notion of product Lévy area requires some control with respect to the operator norm, that is in $L^\infty(\varphi)$ (along (1)), and not only with respect to the $L^2(\varphi)$ -norm (see Section 2 and especially Definition 2.2 for more details on the topology involved in this control).

In the particular case of the free Bm ($q = 0$), the Burkholder-Gundy inequality established by Biane and Speicher in [2, Theorem 3.2.1] immediately gives rise to operator-norm controls on the free Itô integral, which we could readily exploit in [7] to deal with rough paths in the free situation. Unfortunately, and at least for the time being, no similar operator-norm control has been shown for the q -Itô integral when $q \in (0, 1)$. With our rough-path objectives in mind, we will be able to overcome this difficulty though, by resorting to a straightforward $L^\infty(\varphi)$ -construction of a product Lévy area - the latter object being actually much more specific than a general Itô integral. This is the purpose of the forthcoming Section 3 and one of the main results of the paper. Injecting this construction into the general rough-path theory will immediately answer our original issue, that is the derivation of a robust stochastic calculus for the q -Bm.

It is then possible to compare, a posteriori, the resulting rough integral with more familiar q -Itô or q -Stratonovich integrals, through an elementary $L^2(\varphi)$ -analysis and the use of some of the results in [8] (see Section 4). Let us however insist, one more time, on the fact that this sole $L^2(\varphi)$ -analysis would not have been sufficient for the rough-path theory (and the powerful rough-path results) to be applied in this situation.

(ii) In a second part of the paper (starting from Section 5), we will turn to the study of the rich combinatorial machinery governing the behaviour of q -Wiener chaoses. Our main result in this setting is the exhibition of a *full q -Wick product formula*, that is a clear description of the product of any number of multiple integrals (Theorem 6.1). Again, this result must be regarded as a q -extension of a classical

formula, the full Wick formula, that was only known in the two standard situations ($q \in \{0, 1\}$) so far. We thus hope that our analysis can offer a new insight on the transition properties from the commutative to the free case.

To achieve our purpose, we will be led to rephrase the central concept of contraction in terms of q -contraction along (possibly “incomplete”) pairings. Thanks to this representation, many of our arguments can then be conveniently illustrated through basic pictures, and in that sense, our formulation is somehow related to the nice combinatorial approach developed by Nica and Speicher in the free case [14] (of course, the analysis in the general q -Bm situation is far from being as complete as in the specific free case).

We will conclude this study of multiplication in the q -Wiener chaos with a few additional details about the fully-symmetric situation, that is the situation where the integrals involve fully-symmetric kernels. In this case, it turns out that the coefficients in the product formula can be expressed in terms of classical q -combinatorial numbers, making the link with the classical commutative case even more clear (Theorem 7.2). The identification procedure will again give us the opportunity to offer a glimpse on the rich combinatorics associated with the q -Brownian process.

Let us finally point out the fact that, in comparison with the analysis carried out in [3, 8], our presentation throughout the paper will follow a “probabilistic” pattern. In other words, our arguments will essentially rely on the sole law of the q -Bm, that is on the process as given by Definition 1.3. Thus, in the vast majority of our results and proofs, no reference will be made to any particular representation of the process as a map with values in some specific algebra (just as classical probability theory builds upon the law of the Brownian motion and not upon its construction). We consider this fully-stochastic approach to be another advance of the study, which hopefully can make the paper accessible to a large audience. The only reference to some particular representation of the q -Brownian motion (namely its standard representation on the q -Fock space) will occur in Section 4, as a way to compare our rough objects with the constructions of [8], based on the Fock space.

Besides, we have chosen in this study to focus on the case where $q \in [0, 1)$ and introduce the q -Brownian motion as a natural interpolation between the free and the standard Brownian motions. We are aware that the definition of a q -Bm can also be extended to every $q \in (-1, 0)$, that is up to the “anticommutative” situation $q \rightarrow -1$. Let us therefore specify that the positiveness assumption on q will be used in an essential way for our main rough-path construction, that is in the proof of Theorem 3.1. On the contrary, it can be checked that all our arguments and results regarding q -Wiener chaoses (Sections 5-7) are valid for any $q \in (-1, 1)$.

As we already sketched it in the above description of our results, the study is organized as follows. In Section 2, we will recall the general non-commutative rough-path results obtained in [7] and at the core of the present analysis. Section 3 is devoted to the construction of the main object involved in the rough-path procedure, that is a product Lévy area above the q -Bm, while Section 4 focuses on the $L^2(\varphi)$ -comparison of the rough constructions with more standard Itô/Stratonovich definitions. Sections 5, 6 and 7 then deal with the multiplication issue in the q -Wiener chaos, along the above-detailed progression.

2. GENERAL ROUGH-PATH RESULTS IN C^* -ALGEBRAS

Our strategy to develop a robust $L^\infty(\varphi)$ -stochastic calculus for the q -Bm is based on the non-commutative rough-path considerations of [7, Section 4]. Therefore, before we turn to the q -Bm situation and for the sake of completeness, we would like to recall in this section the main results of the theoretical analysis carried out in [7]. This requires first a few brief preliminaries on functional calculus in a C^* -algebra (along the framework of [2]), as well as precisions on the topologies involved in this study. Special emphasis will be put on the cornerstone of the rough-path machinery, the *product Lévy area*, around which the whole integration procedure can be naturally expanded.

Note that the considerations of this section apply to a general C^* -algebra \mathcal{A} , that we fix from now on. In particular, no additional trace operator will be required here. As before, we denote by $\|\cdot\|$ the operator norm on \mathcal{A} , and \mathcal{A}_* will stand for the set of self-adjoint operators in \mathcal{A} . We also fix an arbitrary time horizon $T > 0$ for the whole section.

2.1. Tensor product. Let $\mathcal{A} \otimes \mathcal{A}$ be the algebraic tensor product generated by \mathcal{A} , and just as in [2], denote by \sharp the natural product interaction between \mathcal{A} and $\mathcal{A} \otimes \mathcal{A}$, that is the linear extension of the formula

$$(U_1 \otimes U_2) \sharp X = X \sharp (U_1 \otimes U_2) := U_1 X U_2, \quad \text{for all } U_1, U_2, X \in \mathcal{A}.$$

In a similar way, set, for all $U_1, U_2, U_3, X \in \mathcal{A}$,

$$X \sharp (U_1 \otimes U_2 \otimes U_3) := (U_1 X U_2) \otimes U_3, \quad (U_1 \otimes U_2 \otimes U_3) \sharp X := U_1 \otimes (U_2 X U_3).$$

Our below developments will actually involve the *projective tensor product* $\mathcal{A} \hat{\otimes} \mathcal{A}$ of \mathcal{A} , that is the completion of $\mathcal{A} \otimes \mathcal{A}$ with respect to the norm

$$\|\mathbf{U}\| = \|\mathbf{U}\|_{\mathcal{A} \hat{\otimes} \mathcal{A}} := \inf \sum_i \|U_i\| \|V_i\|,$$

where the infimum is taken over all possible representation $\mathbf{U} = \sum_i U_i \otimes V_i$ of \mathbf{U} . It is readily checked that for all $\mathbf{U} \in \mathcal{A} \otimes \mathcal{A}$ and $X \in \mathcal{A}$, one has $\|\mathbf{U} \sharp X\| \leq \|\mathbf{U}\| \|X\|$, and so the above \sharp -product continuously extends to $\mathcal{A} \hat{\otimes} \mathcal{A}$. These considerations can, of course, be generalized to the n -th projective tensor product $\mathcal{A}^{\hat{\otimes} n}$, $n \geq 1$, and we will still denote by $\|\cdot\|$ the projective tensor norm on $\mathcal{A}^{\hat{\otimes} n}$.

Along the same terminology as in [2], we will call any process with values in $\mathcal{A} \hat{\otimes} \mathcal{A}$, resp. $\mathcal{A} \hat{\otimes} \mathcal{A} \hat{\otimes} \mathcal{A}$, a *biprocess*, resp. a *triprocess*.

2.2. Functional calculus in a C^* -algebra. Following again the presentation of [2], let us introduce the class of functions f defined for every integer $k \geq 0$ by

$$\mathbf{F}_k := \{f : \mathbb{R} \rightarrow \mathbb{C} : f(x) = \int_{\mathbb{R}} e^{i\xi x} \mu_f(d\xi) \text{ with } \int_{\mathbb{R}} |\xi|^i \mu_f(d\xi) < \infty \text{ for every } i \in \{0, \dots, k\}\}, \quad (7)$$

and set, if $f \in \mathbf{F}_k$, $\|f\|_k := \sum_{i=0}^k \int_{\mathbb{R}} |\xi|^i \mu_f(d\xi)$. Then, with all $f \in \mathbf{F}_0$ and $X \in \mathcal{A}_*$, we associate the operator $f(X)$ along the formula

$$f(X) := \int_{\mathbb{R}} e^{i\xi X} \mu_f(d\xi),$$

where the integral in the right-hand side is uniformly convergent in \mathcal{A} . This straightforward operator extension of functional calculus happens to be compatible with Taylor expansions of f , a central ingredient towards the application of rough-path techniques. The following notion of tensor derivatives naturally arises in the procedure (see the subsequent Examples 2.5 and 2.6):

Definition 2.1. For every $f \in \mathbf{F}_1$, resp. $f \in \mathbf{F}_2$, we define the tensor derivative, resp. second tensor derivative, of f by the formula: for every $X \in \mathcal{A}_*$,

$$\partial f(X) := \int_0^1 d\alpha \int_{\mathbb{R}} i\xi [e^{i\alpha\xi X} \otimes e^{i(1-\alpha)\xi X}] \mu_f(d\xi) \in \mathcal{A} \hat{\otimes} \mathcal{A},$$

$$\text{resp. } \partial^2 f(X) := - \iint_{\substack{\alpha, \beta \geq 0 \\ \alpha + \beta \leq 1}} d\alpha d\beta \int_{\mathbb{R}} \xi^2 [e^{i\alpha\xi X} \otimes e^{i\beta\xi X} \otimes e^{i(1-\alpha-\beta)\xi X}] \mu_f(d\xi) \in \mathcal{A} \hat{\otimes} \mathcal{A} \hat{\otimes} \mathcal{A}.$$

2.3. Filtration and Hölder topologies. From now on and for the rest of Section 2, we fix a process $X : [0, T] \rightarrow \mathcal{A}_*$ and assume that X is γ -Hölder regular, that is

$$\sup_{0 \leq s < t \leq T} \frac{\|X_t - X_s\|}{|t - s|^\gamma} < \infty,$$

for some fixed coefficient $\gamma \in (1/3, 1/2)$.

With this process in hand, we denote by $\{\mathcal{A}_t\}_{t \in [0, T]} = \{\mathcal{A}_t^X\}_{t \in [0, T]}$ the *filtration* generated by X , that is, for each $t \in [0, T]$, \mathcal{A}_t stands for the closure (with respect to the operator norm) of the unital subalgebra of \mathcal{A} generated by $\{X_s\}_{0 \leq s \leq t}$.

For any fixed interval $I \subset [0, T]$, a process $Y : I \rightarrow \mathcal{A}$ is said to be *adapted* if for each $t \in I$, $Y_t \in \mathcal{A}_t$. In the same way, a biprocess $\mathbf{U} : [0, T] \rightarrow \mathcal{A} \hat{\otimes} \mathcal{A}$, resp. a triprocess $\mathcal{U} : [0, T] \rightarrow \mathcal{A} \hat{\otimes} \mathcal{A} \hat{\otimes} \mathcal{A}$, is *adapted* if for each $t \in [0, T]$, $\mathbf{U}_t \in \mathcal{A}_t \hat{\otimes} \mathcal{A}_t$, resp. $\mathcal{U}_t \in \mathcal{A}_t \hat{\otimes} \mathcal{A}_t \hat{\otimes} \mathcal{A}_t$.

Let us now briefly recall the topologies involved in the rough-path procedure, as far as time-roughness is concerned (and following Gubinelli's approach [11]). For $V := \mathcal{A}^{\hat{\otimes} n}$ ($n \geq 1$), let $\mathcal{C}_1(I; V)$ be the set of continuous V -valued maps on I , and $\mathcal{C}_2(I; V)$ the set of continuous V -valued maps on the simplex

$\mathcal{S}_2 := \{(s, t) \in I^2 : s \leq t\}$ that vanish on the diagonal. The increments of a path $g \in \mathcal{C}_1(I; V)$ will be denoted by $\delta g_{st} := g_t - g_s$ ($s \leq t$) and for every $\alpha \in (0, 1)$, we define the α -Hölder spaces $\mathcal{C}_1^\alpha(I; V)$, resp. $\mathcal{C}_2^\alpha(I; V)$, as

$$\mathcal{C}_1^\alpha(I; V) := \left\{ h \in \mathcal{C}_1(I; V) : \mathcal{N}[h; \mathcal{C}_1^\alpha(I; V)] := \sup_{s < t \in I} \frac{\|\delta h_{st}\|}{|t - s|^\alpha} < \infty \right\},$$

resp.

$$\mathcal{C}_2^\alpha(I; V) := \left\{ h \in \mathcal{C}_2(I; V) : \mathcal{N}[h; \mathcal{C}_2^\alpha(I; V)] := \sup_{s < t \in I} \frac{\|h_{st}\|}{|t - s|^\alpha} < \infty \right\}.$$

2.4. The product Lévy area. Consider the successive spaces

$$\mathcal{L}_T(\mathcal{A}_\rightarrow) := \{L = (L_{st})_{0 \leq s < t \leq T} : L_{st} \in \mathcal{L}(\mathcal{A}_s \hat{\otimes} \mathcal{A}_t, \mathcal{A})\},$$

$$\mathcal{L}_T(\mathcal{A}_\rightarrow) := \{L = (L_{st})_{0 \leq s < t \leq T} : L_{st} \in \mathcal{L}(\mathcal{A}_s \hat{\otimes} \mathcal{A}_t, \mathcal{A}_t)\},$$

and for every $\lambda \in [0, 1]$, denote by $\mathcal{C}_2^\lambda(\mathcal{L}_T(\mathcal{A}_\rightarrow))$, resp. $\mathcal{C}_2^\lambda(\mathcal{L}_T(\mathcal{A}_\rightarrow))$, the set of elements $L \in \mathcal{L}_T(\mathcal{A}_\rightarrow)$, resp. $L \in \mathcal{L}_T(\mathcal{A}_\rightarrow)$, for which the following quantity is finite:

$$\mathcal{N}[L; \mathcal{C}_2^\lambda(\mathcal{L}_T(\mathcal{A}_\rightarrow))] := \sup_{\substack{s < t \in [0, T] \\ \mathbf{U} \in \mathcal{A}_s \hat{\otimes} \mathcal{A}_t, \mathbf{U} \neq 0}} \frac{\|L_{st}[\mathbf{U}]\|}{|t - s|^\lambda \|\mathbf{U}\|}. \quad (8)$$

At this point, recall that we have fixed a γ -Hölder process $X : [0, T] \rightarrow \mathcal{A}_*$ ($\gamma \in (1/3, 1/2)$) for the whole section 2.

Definition 2.2. We call product Lévy area above X any process \mathbb{X}^2 such that:

- (i) (2γ -roughness) $\mathbb{X}^2 \in \mathcal{C}_2^{2\gamma}(\mathcal{L}_T(\mathcal{A}_\rightarrow))$,
- (ii) (Product Chen identity) For all $s < u < t$ and $\mathbf{U} \in \mathcal{A}_s \hat{\otimes} \mathcal{A}_t$,

$$\mathbb{X}_{st}^2[\mathbf{U}] - \mathbb{X}_{su}^2[\mathbf{U}] - \mathbb{X}_{ut}^2[\mathbf{U}] = (\mathbf{U} \sharp \delta X_{su}) \delta X_{ut}. \quad (9)$$

Remark 2.3. Recall that Definition 2.2 is derived from the theoretical analysis performed in [7, Section 4] with equation (6) in mind. At some heuristic level, and following the classical rough-path approach, the notion of product Lévy area must be seen as some abstract version of the iterated integral

$$\mathbb{X}_{st}^2[\mathbf{U}] = \int_s^t (\mathbf{U} \sharp \delta X_{su}) dX_u, \quad (10)$$

noting that definition of this integral is not clear a priori for a non-differentiable process X . As pointed out in [7], the above notion of “Lévy area” is specifically designed to handle the non-commutative algebra dynamics of (6), and it offers a much more efficient approach than general rough-path theory based on “tensor” Lévy areas (the object considered in [13]). In a commutative setting (i.e., if \mathcal{A} were a commutative algebra), the basic process $\mathbb{A}_{st}(\mathbf{U}) := \frac{1}{2}(\mathbf{U} \sharp \delta X_{st}) \delta X_{st}$ would immediately provide us with such a product Lévy area. In the general (non-commutative) situation though, this path only satisfies

$$\mathbb{A}_{st}[\mathbf{U}] - \mathbb{A}_{su}[\mathbf{U}] - \mathbb{A}_{ut}[\mathbf{U}] = \frac{1}{2}[(\mathbf{U} \sharp \delta X_{su}) \delta X_{ut} + (\mathbf{U} \sharp \delta X_{ut}) \delta X_{su}],$$

so that \mathbb{A} may not meet the product-Chen condition (ii), making Definition 2.2 undoubtedly relevant.

2.5. Controlled (bi)processes and integration. A second ingredient in the rough-path machinery (in addition to a “Lévy area”) consists in the identification of a suitable class of integrands for the future rough integral with respect to X . The following definition naturally arises in this setting:

Definition 2.4. Given a time interval $I \subset [0, T]$, we call adapted controlled process, resp. biprocess, on I any adapted process $Y \in \mathcal{C}_1^\gamma(I; \mathcal{A})$, resp. biprocess $\mathbf{U} \in \mathcal{C}_1^\gamma(I; \mathcal{A} \hat{\otimes} \mathcal{A})$, with increments of the form

$$(\delta Y)_{st} = \mathbf{Y}_s^X \sharp (\delta X)_{st} + Y_{st}^b, \quad s < t \in I, \quad (11)$$

resp.

$$(\delta \mathbf{U})_{st} = (\delta X)_{st} \sharp \mathcal{U}_s^{X,1} + \mathcal{U}_s^{X,2} \sharp (\delta X)_{st} + \mathbf{U}_{st}^b, \quad s < t \in I, \quad (12)$$

for some adapted biprocess $\mathbf{Y}^X \in \mathcal{C}_1^\gamma(I; \mathcal{A} \hat{\otimes} \mathcal{A})$, resp. adapted triprocesses $\mathcal{U}^{X,1}, \mathcal{U}^{X,2} \in \mathcal{C}_1^\gamma(I; \mathcal{A} \hat{\otimes}^3)$, and $Y^b \in \mathcal{C}_2^{2\gamma}(I; \mathcal{A})$, resp. $\mathbf{U}^b \in \mathcal{C}_2^{2\gamma}(I; \mathcal{A} \hat{\otimes} \mathcal{A})$. We denote by $\mathcal{Q}_X(I)$, resp. $\mathbf{Q}_X(I)$, the space of adapted

controlled processes, resp. biprocesses, on I , and finally we define $\mathcal{Q}_X^*(I)$ as the subspace of controlled processes $Y \in \mathcal{Q}_X(I)$ for which one has both $Y_s^* = Y_s$ and $(Y_s^X)^* = Y_s^X$ for every $s \in I$.

Example 2.5. If $f, g \in \mathbf{F}_2$ and $Y \in \mathcal{Q}_X^*(I)$ with decomposition (11), then $\mathbf{U} := f(Y) \otimes g(Y) \in \mathbf{Q}_X(I)$ with

$$\mathcal{U}_s^{X,1} := [\partial f(Y_s) \mathbf{Y}_s^X] \otimes g(Y_s) \quad , \quad \mathcal{U}_s^{X,2} = f(Y_s) \otimes [\partial g(Y_s) \mathbf{Y}_s^X] .$$

Example 2.6. If $f \in \mathbf{F}_3$, then $\mathbf{U} := \partial f(X) \in \mathbf{Q}_X([0, T])$ with $\mathcal{U}_s^{X,1} = \mathcal{U}_s^{X,2} = \partial^2 f(X_s)$.

We are finally in a position to recall the definition of the *rough integral* with respect to X , which can be expressed (among other ways) as the limit of “corrected Riemann sums”:

Proposition 2.7. [7, Proposition 4.12] *Assume that we are given a product Lévy area \mathbb{X}^2 above X , in the sense of Definition 2.2, as well as a time interval $I = [\ell_1, \ell_2] \subset [0, T]$. Then for every $\mathbf{U} \in \mathbf{Q}_X(I)$ with decomposition (12), all $s < t \in I$ and every subdivision $D_{st} = \{t_0 = s < t_1 < \dots < t_n = t\}$ of $[s, t]$ with mesh $|D_{st}|$ tending to 0, the corrected Riemann sum*

$$\sum_{t_i \in D_{st}} \left\{ \mathbf{U}_{t_i} \sharp (\delta X)_{t_i t_{i+1}} + [\mathbb{X}_{t_i t_{i+1}}^2 \times Id](\mathcal{U}_{t_i}^{X,1}) + [Id \times \mathbb{X}_{t_i t_{i+1}}^{2,*}](\mathcal{U}_{t_i}^{X,2}) \right\}$$

converges in \mathcal{A} as $|D_{st}| \rightarrow 0$. We call the limit the *rough integral* (from s to t) of \mathbf{U} against $\mathbb{X} := (X, \mathbb{X}^2)$, and we denote it by $\int_s^t \mathbf{U}_u \sharp d\mathbb{X}_u$. This construction satisfies the two following properties:

- (Consistency) If X is a differentiable process in \mathcal{A} and \mathbb{X}^2 is understood in the classical Lebesgue sense (that is, as in (10)), then $\int_s^t \mathbf{U}_u \sharp d\mathbb{X}_u$ coincides with the classical Lebesgue integral $\int_s^t [\mathbf{U}_u \sharp X'_u] du$;
- (Stability) For every $A \in \mathcal{A}$, there exists a unique process $Z \in \mathcal{Q}_X(I)$ such that $Z_{\ell_1} = A$ and $(\delta Z)_{st} = \int_s^t \mathbf{U}_u \sharp d\mathbb{X}_u$ for all $s < t \in I$.

Theorem 2.8. [7, Theorem 4.15] *Assume that we are given a product Lévy area \mathbb{X}^2 above X . Let $f = (f_1, \dots, f_m) \in \mathbf{F}_3^m$, $g = (g_1^*, \dots, g_m^*)$ or (f_m^*, \dots, f_1^*) , and fix $A \in \mathcal{A}_*$. Then the equation*

$$Y_0 = A \quad , \quad (\delta Y)_{st} = \sum_{i=1}^m \int_s^t f_i(Y_u) d\mathbb{X}_u g_i(Y_u) \quad , \quad s < t \in [0, T] \quad , \quad (13)$$

interpreted with Proposition 2.7, admits a unique solution $Y \in \mathcal{Q}_X^*([0, T])$.

2.6. Approximation results. Another advantage of the rough-path approach - beyond its consistency and stability properties - lies in the continuity of the constructions with respect to the driving (rough) path. In this non-commutative setting, and following the approach of [7], the phenomenon can be illustrated through several “Wong-Zakai-type” approximation results, which we propose to briefly review here. To this end, for every sequence of partitions (D^n) of $[0, T]$ with mesh tending to zero, denote by $\{X_t^n\}_{t \in [0, T]} = \{X_t^{D^n}\}_{t \in [0, T]}$ the sequence of linear interpolations of X along D^n , i.e., if $D^n := \{0 = t_0 < t_1 < \dots < t_k = T\}$,

$$X_t^n := X_{t_i} + \frac{t - t_i}{t_{i+1} - t_i} \delta X_{t_i t_{i+1}} \quad \text{for } t \in [t_i, t_{i+1}] .$$

Then consider the sequence of approximated product Lévy areas defined for every $\mathbf{U} \in \mathcal{A} \hat{\otimes} \mathcal{A}$ as

$$\mathbb{X}_{st}^{2,n}[\mathbf{U}] = \mathbb{X}_{st}^{2,D^n}[\mathbf{U}] := \int_s^t (\mathbf{U} \sharp \delta X_{su}^n) dX_u^n \quad , \quad s < t \in [0, T] \quad , \quad (14)$$

where the integral is understood in the classical Lebesgue sense.

Proposition 2.9. [7, Proposition 4.16] *Assume that there exists a product Lévy area \mathbb{X}^2 above X such that, as n tends to infinity,*

$$\mathcal{N}[X^n - X; \mathcal{C}_1^\gamma([0, T]; \mathcal{A})] \rightarrow 0 \quad \text{and} \quad \mathcal{N}[\mathbb{X}^{2,n} - \mathbb{X}^2; \mathcal{C}_2^{2\gamma}(\mathcal{L}_T(\mathcal{A}_-))] \rightarrow 0 . \quad (15)$$

Then for all $f, g \in \mathbf{F}_3$, it holds that

$$\int_s^t f(X_u^n) dX_u^n g(X_u^n) \xrightarrow{n \rightarrow \infty} \int_s^t f(X_u) d\mathbb{X}_u g(X_u) \quad \text{in } \mathcal{C}_2^\gamma([0, T]; \mathcal{A}) \quad , \quad (16)$$

where the integral in the limit is interpreted with Proposition 2.7. Similarly, for all $f \in \mathbf{F}_3$, one has

$$\int \partial f(X_u^n) \sharp dX_u^n \xrightarrow{n \rightarrow \infty} \int \partial f(X_u) \sharp d\mathbb{X}_u \quad \text{in } \mathcal{C}_2^\gamma([0, T]; \mathcal{A}), \quad (17)$$

which immediately yields Itô's formula: for all $s < t \in [0, T]$,

$$\delta f(X)_{st} = \int_s^t \partial f(X_u) \sharp d\mathbb{X}_u. \quad (18)$$

Finally, for some fixed $f = (f_1, \dots, f_m) \in \mathbf{F}_3^m$ and $g = (f_1^*, \dots, f_m^*)$ (or $g := (f_m^*, \dots, f_1^*)$), let us denote by $Y^n = Y^{D^n}$ the solution of the classical Lebesgue equation on $[0, T]$

$$Y_0^n = A \in \mathcal{A}_* \quad , \quad dY_t^n = \sum_{i=1}^m f_i(Y_t^n) dX_t^n g_i(Y_t^n).$$

Theorem 2.10. [7, Theorem 4.17] *Under the assumptions of Proposition 2.9, one has $Y^n \xrightarrow{n \rightarrow \infty} Y$ in $\mathcal{C}_1^\gamma([0, T]; \mathcal{A})$, where Y is the solution of (13) given by Theorem 2.8.*

3. A PRODUCT LÉVY AREA ABOVE THE q -BROWNIAN MOTION

We go back here to the q -Bm setting described in Section 1. Namely, we fix $q \in [0, 1]$ and consider a q -Brownian motion $(X_t)_{t \geq 0}$ in some non-commutative probability space (\mathcal{A}, φ) . With the developments of the previous section in mind, the route towards an efficient operator-norm calculus for X is now clear: we need to exhibit a product Lévy area above X , in the sense of Definition 2.2. Our main result thus reads as follows:

Theorem 3.1. *Denote by $\{X_t^n\}_{t \geq 0}$ the linear interpolation of X along the dyadic partition $D^n := \{t_i^n, i \geq 0\}$, $t_i^n := \frac{i}{2^n}$. Then there exists a product Lévy area $\mathbb{X}^{2,S}$ above X , in the sense of Definition 2.2, such that for every $T > 0$ and every $0 < \gamma < 1/2$, one has*

$$X^n \rightarrow X \quad \text{in } \mathcal{C}_1^\gamma([0, T]; \mathcal{A}) \quad \text{and} \quad \mathbb{X}^{2,n} \rightarrow \mathbb{X}^{2,S} \quad \text{in } \mathcal{C}_2^{2\gamma}(\mathcal{L}_T(\mathcal{A}_\rightarrow)), \quad (19)$$

where $\mathbb{X}^{2,n}$ is defined by (14). We call $\mathbb{X}^{2,S}$ the Stratonovich product Lévy area above X .

Based on this result, the conclusions of Proposition 2.7, Theorem 2.8, Proposition 2.9 and Theorem 2.10 can all be applied to the q -Brownian motion, with limits understood as rough integrals with respect to the “product rough path” $\mathbb{X}^S := (X, \mathbb{X}^{2,S})$. The Stratonovich terminology is here used as a reference to the classical commutative situation, where the (almost sure) limit of the sequence of approximated Lévy areas would indeed coincide with the Stratonovich iterated integral (see also Corollary 4.10 for another justification of this terminology).

Before we turn to the proof of Theorem 3.1, let us recall that the whole difficulty in constructing a stochastic integral with respect to the general q -Bm, in comparison with the free ($q = 0$) or the commutative ($q \rightarrow 1$) cases, lies in the absence of any satisfying “ q -freeness” property for the increments of the process when $q \in (0, 1)$ (as reported by Speicher in [16]). For instance, if $s < u < t$,

$$\varphi((X_u - X_s)(X_t - X_u)(X_u - X_s)(X_t - X_u)) = q \varphi((X_u - X_s)^2) \varphi((X_t - X_u)^2) = q |u - s| |t - u|,$$

which shows that, for $q \neq 0$, the disjoint increments of a q -Brownian motion $\{X_t\}_{t \geq 0}$ are indeed not freely independent (in the sense of [7, Definition 2.6]), making most of the arguments of [2] unexploitable in this situation.

This being said, we can still rely here on the basic fact that for all $q \in [0, 1]$, $\varphi((X_u - X_s)(X_t - X_u)) = 0$. Together with Formula (3), this very weak “freeness” property of the increments will somehow be sufficient for our purpose, the construction of a product Lévy area being much more specific than the construction of a general stochastic integral (along Itô's standard procedure).

The proof of Theorem 3.1 will also appeal to the two following elementary lemmas. The first one (whose proof follows immediately from (3)) is related to the linear stability of q -Gaussian families:

Lemma 3.2. *For any fixed $q \in [0, 1]$, let $Y := \{Y_1, \dots, Y_d\}$ be a q -Gaussian vector in some non-commutative probability space (\mathcal{A}, φ) , and consider a real-valued $(d \times m)$ -matrix Λ . Then $Z := \Lambda Y$ is also a q -Gaussian vector in (\mathcal{A}, φ) .*

We will also need the following general topology property on the space accommodating any Lévy area:

Lemma 3.3. *The space $\mathcal{C}_2^\lambda(\mathcal{L}_T(\mathcal{A}_\perp))$, endowed with the norm (8), is complete.*

Proof. Although the arguments are classical, let us provide a few details here, since the $\mathcal{C}_2^\lambda(\mathcal{L}_T(\mathcal{A}_\perp))$ -structure is not exactly standard. Consider a Cauchy sequence L^n in $\mathcal{C}_2^\lambda(\mathcal{L}_T(\mathcal{A}_\perp))$. For every fixed $s \in [0, T]$, the sequence L_s^n defines a Cauchy sequence in the space $L^\infty([s, T]; \mathcal{L}(\mathcal{A}_s \hat{\otimes} \mathcal{A}_s, \mathcal{A}))$ of bounded functions on $[s, T]$ (with values in $\mathcal{L}(\mathcal{A}_s \hat{\otimes} \mathcal{A}_s, \mathcal{A})$), endowed with the uniform norm. Therefore it converges in the latter space to some function $L_{s,\cdot}$. The fact that the so-defined family $\{L_{st}\}_{s < t}$ belongs to $\mathcal{C}_2^\lambda(\mathcal{L}_T(\mathcal{A}_\perp))$ is an immediate consequence of the boundedness of L^n in $\mathcal{C}_2^\lambda(\mathcal{L}_T(\mathcal{A}_\perp))$. Finally, given $\varepsilon > 0$ and for all fixed $s < t$, we know that there exists $M_{\varepsilon,s,t} \geq 0$ such that for all $m \geq M_{\varepsilon,s,t}$, $\|L_{st}^m - L_{st}\|_{\mathcal{L}(\mathcal{A}_s \hat{\otimes} \mathcal{A}_s, \mathcal{A})} \leq \frac{\varepsilon}{2}|t - s|^\lambda$. On the other hand, there exists $N_\varepsilon \geq 0$ such that for all $n, m \geq N_\varepsilon$ and all $s < t$, $\|L_{st}^n - L_{st}^m\|_{\mathcal{L}(\mathcal{A}_s \hat{\otimes} \mathcal{A}_s, \mathcal{A})} \leq \frac{\varepsilon}{2}|t - s|^\lambda$. Therefore, for all $n \geq N_\varepsilon$ and all $s < t$, we get that for $m := \max(N_\varepsilon, M_{\varepsilon,s,t})$,

$$\|L_{st}^n - L_{st}\|_{\mathcal{L}(\mathcal{A}_s \hat{\otimes} \mathcal{A}_s, \mathcal{A})} \leq \|L_{st}^n - L_{st}^m\|_{\mathcal{L}(\mathcal{A}_s \hat{\otimes} \mathcal{A}_s, \mathcal{A})} + \|L_{st} - L_{st}^m\|_{\mathcal{L}(\mathcal{A}_s \hat{\otimes} \mathcal{A}_s, \mathcal{A})} \leq \varepsilon|t - s|^\lambda,$$

and so $L^n \rightarrow L$ in $\mathcal{C}_2^\lambda(\mathcal{L}_T(\mathcal{A}_\perp))$, which achieves to prove that the latter space is complete. \square

Proof of Theorem 3.1. Throughout the proof, we will denote by $A \lesssim B$ any bound of the form $A \leq cB$, where c is a universal constant independent from the parameters under consideration. The first-order convergence statement in (19) is a straightforward consequence of the $1/2$ -Hölder regularity of X . In fact, using (5), it can be checked that for all $n \geq 0$ and $s < t$,

$$\|\delta X_{st}^n\| \lesssim \|X_1\| |t - s|^{1/2} \quad \text{and} \quad \|\delta(X^n - X)_{st}\| \lesssim \|X_1\| |t - s|^{\gamma 2^{-n(1/2-\gamma)}}. \quad (20)$$

Let us turn to the second-order convergence and to this end, fix $n \geq 0$ and $s < t$ such that $t_k^n \leq s < t_{k+1}^n$, $t_\ell^n \leq t < t_{\ell+1}^n$, with $k \leq \ell$. If $|\ell - k| \leq 1$, or in other words if $|t - s| \leq 2^{-n+1}$, the expected bound can be readily derived from the first estimate in (20), that is for every $\mathbf{U} \in \mathcal{A}_s \hat{\otimes} \mathcal{A}_s$, we get from (20)

$$\|\mathbb{X}_{st}^{2,n+1}[\mathbf{U}] - \mathbb{X}_{st}^{2,n}[\mathbf{U}]\| \leq \|\mathbb{X}_{st}^{2,n+1}[\mathbf{U}]\| + \|\mathbb{X}_{st}^{2,n}[\mathbf{U}]\| \lesssim \|X_1\|^2 |t - s|^{2\gamma 2^{-n(1/2-\gamma)}}.$$

Assume from now on that $\ell \geq k + 2$ and in this case consider the decomposition, for every $\mathbf{U} \in \mathcal{A}_s \hat{\otimes} \mathcal{A}_s$,

$$\begin{aligned} & \mathbb{X}_{st}^{2,n+1}[\mathbf{U}] - \mathbb{X}_{st}^{2,n}[\mathbf{U}] \\ &= \left[\int_{t_{k+1}^n}^{t_\ell^n} (\mathbf{U} \# \delta X_{t_{k+1}^n u}^{n+1}) dX_u^{n+1} - \int_{t_{k+1}^n}^{t_\ell^n} (\mathbf{U} \# \delta X_{t_{k+1}^n u}^n) dX_u^n \right] \\ &+ \left[\int_s^{t_{k+1}^n} (\mathbf{U} \# \delta X_{su}^{n+1}) dX_u^{n+1} + \int_{t_\ell^n}^t (\mathbf{U} \# \delta X_{su}^{n+1}) dX_u^{n+1} - \int_s^{t_{k+1}^n} (\mathbf{U} \# \delta X_{su}^n) dX_u^n \right. \\ &\quad \left. - \int_{t_\ell^n}^t (\mathbf{U} \# \delta X_{su}^n) dX_u^n \right] \\ &+ \left[\int_{t_{k+1}^n}^{t_\ell^n} (\mathbf{U} \# \delta X_{st_{k+1}^n}^{n+1}) dX_u^{n+1} - \int_{t_{k+1}^n}^{t_\ell^n} (\mathbf{U} \# \delta X_{st_{k+1}^n}^n) dX_u^n \right]. \end{aligned} \quad (21)$$

The “boundary” integrals within the second and third brackets can again be bounded individually using the first estimate in (20) only. For instance,

$$\begin{aligned} \left\| \int_{t_\ell^n}^t (\mathbf{U} \# \delta X_{su}^{n+1}) dX_u^{n+1} \right\| &\lesssim \|X_1\| \|\mathbf{U}\| \left[\mathbf{1}_{\{t_{2\ell}^{n+1} \leq t < t_{2\ell+1}^{n+1}\}} \int_{t_\ell^n}^t |s - u|^{1/2} (2^{n+1} \|\delta X_{t_{2\ell}^{n+1} t_{2\ell+1}^{n+1}}\|) \right. \\ &\quad + \mathbf{1}_{\{t_{2\ell+1}^{n+1} \leq t < t_{2\ell+2}^{n+1}\}} \int_{t_{2\ell}^{n+1}}^{t_{2\ell+1}^{n+1}} |s - u|^{1/2} (2^{n+1} \|\delta X_{t_{2\ell}^{n+1} t_{2\ell+1}^{n+1}}\|) \\ &\quad \left. + \mathbf{1}_{\{t_{2\ell+1}^{n+1} \leq t < t_{2\ell+2}^{n+1}\}} \int_{t_{2\ell+1}^{n+1}}^t |s - u|^{1/2} (2^{n+1} \|\delta X_{t_{2\ell+1}^{n+1} t_{2\ell+2}^{n+1}}\|) \right] \\ &\lesssim \|X_1\|^2 \|\mathbf{U}\| |t - s|^{2\gamma 2^{-n(1/2-\gamma)}}. \end{aligned}$$

Therefore, we only have to focus on the first bracket in decomposition (21). In fact, using only the very definition of the approximation, it can be checked that this term is also equal to

$$\begin{aligned} & \int_{t_{k+1}^n}^{t_\ell^n} (\mathbf{U} \# \delta X_{t_{k+1}^n}^{n+1} u) dX_u^{n+1} - \int_{t_{k+1}^n}^{t_\ell^n} (\mathbf{U} \# \delta X_{t_{k+1}^n}^n u) dX_u^n \\ &= \frac{1}{2} \sum_{i=k+1}^{\ell-1} \left[(\mathbf{U} \# (\delta X)_{t_{2i}^{n+1} t_{2i+1}^{n+1}}) (\delta X)_{t_{2i+1}^{n+1} t_{2i+2}^{n+1}} - (\mathbf{U} \# (\delta X)_{t_{2i+1}^{n+1} t_{2i+2}^{n+1}}) (\delta X)_{t_{2i}^{n+1} t_{2i+1}^{n+1}} \right]. \end{aligned} \quad (22)$$

Let us bound the two sums

$$S_{st}^{1,n}[\mathbf{U}] := \sum_{i=k+1}^{\ell-1} \left(\mathbf{U} \# (\delta X)_{t_{2i}^{n+1} t_{2i+1}^{n+1}} \right) (\delta X)_{t_{2i+1}^{n+1} t_{2i+2}^{n+1}}, \quad S_{st}^{2,n}[\mathbf{U}] := \sum_{i=k+1}^{\ell-1} \left(\mathbf{U} \# (\delta X)_{t_{2i+1}^{n+1} t_{2i+2}^{n+1}} \right) (\delta X)_{t_{2i}^{n+1} t_{2i+1}^{n+1}}$$

separately.

Consider first the case where $\mathbf{U} = \sum_{j=1}^o U_j \otimes V_j$, with $U_j := X_{s_1^j} \cdots X_{s_{m_j}^j}$, $V_j := X_{s_{m_j+1}^j} \cdots X_{s_{m_j+p_j}^j}$, and $s_p^j \leq s$ for all j, p . Besides, let us set $Y_i = Y_{i,n} := (\delta X)_{t_i^{n+1} t_{i+1}^{n+1}}$. With these notations, and for every $r \geq 1$, we have

$$\begin{aligned} \varphi(|S_{st}^{1,n}[\mathbf{U}]|^{2r}) &= \varphi \left(\left[\left(\sum_{i_1} \sum_{j_1} U_{j_1} Y_{2i_1} V_{j_1} Y_{2i_1+1} \right) \left(\sum_{i_2} \sum_{j_2} U_{j_2} Y_{2i_2} V_{j_2} Y_{2i_2+1} \right)^* \right]^r \right) \\ &= \sum_{i_1, \dots, i_{2r}} \sum_{j_1, \dots, j_{2r}} \varphi \left([U_{j_1} Y_{2i_1} V_{j_1} Y_{2i_1+1} Y_{2i_2+1} V_{j_2}^* Y_{2i_2} U_{j_2}^*] \cdots \right. \\ &\quad \left. [U_{j_{2r-1}} Y_{2i_{2r-1}} V_{j_{2r-1}} Y_{2i_{2r-1}+1} Y_{2i_{2r}+1} V_{j_{2r}}^* Y_{2i_{2r}} U_{j_{2r}}^*] \right], \end{aligned} \quad (23)$$

where each index i runs over $\{k+1, \dots, \ell-1\}$ and each index j runs over $\{1, \dots, o\}$. At this point, observe that for all fixed $\mathbf{i} := (i_1, \dots, i_{2r})$ and $\mathbf{j} := (j_1, \dots, j_{2r})$, the family

$$\{X_{s_1^j}, \dots, X_{s_{m_j+p_j}^j}, Y_{2i}, Y_{2i+1}, i \in \{i_1, \dots, i_{2r}\}, j \in \{j_1, \dots, j_{2r}\}\}$$

is a q -Gaussian family (due to Lemma 3.2) and accordingly the associated joint moments obey Formula (3). Besides, we have trivially

$$\varphi(Y_{2i_a} Y_{2i_b+1}) = 0 \quad , \quad \varphi(Y_{2i_a} Y_{2i_b}) = \varphi(Y_{2i_a+1} Y_{2i_b+1}) = \mathbf{1}_{\{i_a=i_b\}} 2^{-(n+1)} \varphi(|X_1|^2)$$

and

$$\varphi(Y_{2i} X_{s_a^j}) = \varphi(Y_{2i+1} X_{s_a^j}) = 0.$$

Using these basic observations and going back to (23), it is clear that, when applying Formula (3) to the expectation in (23), we can restrict the sum to the set of pairings $\pi \in \mathcal{P}_2(\{1, \dots, N_r\})$ ($N_r := 2[(m_{j_1} + p_{j_1}) + \dots + (m_{j_{2r}} + p_{j_{2r}})] + 8r$) that decompose - in a unique way - as a combination of three sub-pairings, namely: 1) a pairing $\pi^1 \in \mathcal{P}_2(\{1, \dots, 2r\})$ that connects the random variables $\{Y_{2i}\}$ to each other; 2) a pairing $\pi^2 \in \mathcal{P}_2(\{1, \dots, 2r\})$ that connects the random variables $\{Y_{2i+1}\}$ to each other; 3) a pairing $\pi^3 \in \mathcal{P}_2(\{1, \dots, N_r'\})$ ($N_r' := 2[(m_{j_1} + p_{j_1}) + \dots + (m_{j_{2r}} + p_{j_{2r}})]$) that connects the random variables $\{X_{s_i^j}\}$ to each other. Moreover, with this decomposition in mind, one has clearly

$$\text{Cr}(\pi) \geq \text{Cr}(\pi^1) + \text{Cr}(\pi^2) + \text{Cr}(\pi^3).$$

Consequently, it holds that for all fixed $\mathbf{i} := (i_1, \dots, i_{2r})$ and $\mathbf{j} := (j_1, \dots, j_{2r})$,

$$\begin{aligned}
& \left| \varphi \left([U_{j_1} Y_{2i_1} V_{j_1} Y_{2i_1+1} Y_{2i_2+1} V_{j_2}^* Y_{2i_2} U_{j_2}^*] \cdots [U_{j_{2r-1}} Y_{2i_{2r-1}} V_{j_{2r-1}} Y_{2i_{2r-1}+1} Y_{2i_{2r}+1} V_{j_{2r}}^* Y_{2i_{2r}} U_{j_{2r}}^*] \right) \right| \\
& \leq \sum_{\substack{\pi^1, \pi^2 \in \mathcal{P}_2(\{1, \dots, 2r\}) \\ \pi^3 \in \mathcal{P}_2(\{1, \dots, N'_r\})}} q^{\text{Cr}(\pi^1) + \text{Cr}(\pi^2) + \text{Cr}(\pi^3)} \\
& \quad \prod_{\{a,b\} \in \pi^1} \varphi(Y_{2i_a} Y_{2i_b}) \mathbf{1}_{\{i_a=i_b\}} \prod_{\{c,d\} \in \pi^2} \varphi(Y_{2i_c+1} Y_{2i_d+1}) \mathbf{1}_{\{i_c=i_d\}} \prod_{\{e,f\} \in \pi^3} \varphi(Z_e^{\mathbf{j}} Z_f^{\mathbf{j}}) \\
& \leq 2^{-2r(n+1)} \varphi(|X_1|^2)^{2r} \left(\sum_{\pi^1 \in \mathcal{P}_2(\{1, \dots, 2r\})} q^{\text{Cr}(\pi^1)} \prod_{\{a,b\} \in \pi^1} \mathbf{1}_{\{i_a=i_b\}} \right) \\
& \quad \left(\sum_{\pi^2 \in \mathcal{P}_2(\{1, \dots, 2r\})} q^{\text{Cr}(\pi^2)} \prod_{\{c,d\} \in \pi^2} \mathbf{1}_{\{i_c=i_d\}} \right) \left(\sum_{\pi^3 \in \mathcal{P}_2(\{1, \dots, N'_r\})} q^{\text{Cr}(\pi^3)} \prod_{\{e,f\} \in \pi^3} \varphi(Z_e^{\mathbf{j}} Z_f^{\mathbf{j}}) \right),
\end{aligned}$$

where $Z^{\mathbf{j}}$ stands for the natural reordering of the variables $\{X_{s_m^j}\}$, namely for all $a \in \{1, \dots, 2r\}$ and $b \in \{1, \dots, m_{j_a} + p_{j_a}\}$,

$$Z_{2[(m_{j_1}+p_{j_1})+\dots+(m_{j_{a-1}}+p_{j_{a-1}})]+b}^{\mathbf{j}} = Z_{2[(m_{j_1}+p_{j_1})+\dots+(m_{j_{a-1}}+p_{j_{a-1}})]+[2(m_{j_a}+p_{j_a})-b]}^{\mathbf{j}} := X_{s_b^{j_a}}.$$

As a result, the double sum in (23) is bounded by

$$\begin{aligned}
& 2^{-2rn} \varphi(|X_1|^2)^{2r} \left(\sum_{\pi^1 \in \mathcal{P}_2(\{1, \dots, 2r\})} q^{\text{Cr}(\pi^1)} \sum_{i_1, \dots, i_{2r}=k+1}^{\ell-1} \prod_{\{a,b\} \in \pi^1} \mathbf{1}_{\{i_a=i_b\}} \right) \left(\sum_{\pi^2 \in \mathcal{P}_2(\{1, \dots, 2r\})} q^{\text{Cr}(\pi^2)} \right) \\
& \quad \sum_{j_1, \dots, j_{2r}=1}^o \left(\sum_{\pi^3 \in \mathcal{P}_2(\{1, \dots, N'_r\})} q^{\text{Cr}(\pi^3)} \prod_{\{e,f\} \in \pi^3} \varphi(Z_e^{\mathbf{j}} Z_f^{\mathbf{j}}) \right). \quad (24)
\end{aligned}$$

Now observe that the last sum in (24) actually corresponds to

$$\sum_{j_1, \dots, j_{2r}=1}^o \left(\sum_{\pi^3 \in \mathcal{P}_2(\{1, \dots, N'_r\})} q^{\text{Cr}(\pi^3)} \prod_{\{e,f\} \in \pi^3} \varphi(Z_e^{\mathbf{j}} Z_f^{\mathbf{j}}) \right) = \varphi \left(\left| \sum_{j=1}^o U_j V_j \right|^{2r} \right), \quad (25)$$

and for every fixed $\pi^1 \in \mathcal{P}_2(\{1, \dots, 2r\})$,

$$\begin{aligned}
& \sum_{i_1, \dots, i_{2r}=k+1}^{\ell-1} \prod_{\{a,b\} \in \pi^1} \mathbf{1}_{\{i_a=i_b\}} \\
& = \left(\sum_{i_1, \dots, i_{2r}=k+1}^{\ell-1} \prod_{\{a,b\} \in \pi^1} \mathbf{1}_{\{i_a=i_b\}} \right)^{2(1-2\gamma)} \left(\sum_{i_1, \dots, i_{2r}=k+1}^{\ell-1} \prod_{\{a,b\} \in \pi^1} \mathbf{1}_{\{i_a=i_b\}} \right)^{4\gamma-1} \\
& \leq (\ell - (k+1))^{2(1-2\gamma)r} (\ell - (k+1))^{2r(4\gamma-1)} = |t_\ell^n - t_{k+1}^n|^{4r\gamma} 2^{4r\gamma n} \leq |t - s|^{4r\gamma} 2^{4r\gamma n}. \quad (26)
\end{aligned}$$

By injecting (25) and (26) into (24), we end up with the estimate

$$\begin{aligned}
\varphi(|S_{st}^{1,n}[\mathbf{U}]|^{2r}) & \leq |t - s|^{4r\gamma} 2^{-2r(1-2\gamma)n} \varphi(|X_1|^2)^{2r} \left(\sum_{\pi \in \mathcal{P}_2(\{1, \dots, 2r\})} q^{\text{Cr}(\pi)} \right)^2 \varphi \left(\left| \sum_{j=1}^o U_j V_j \right|^{2r} \right) \\
& \leq |t - s|^{4r\gamma} 2^{-2r(1-2\gamma)n} \varphi(|X_1|^2)^{2r} \varphi(|X_1|^{2r})^2 \varphi \left(\left| \sum_{j=1}^o U_j V_j \right|^{2r} \right),
\end{aligned}$$

and so

$$\begin{aligned}
\varphi(|S_{st}^{1,n}[\mathbf{U}]|^{2r})^{1/2r} &\leq |t-s|^{2\gamma} 2^{-(1-2\gamma)n} \varphi(|X_1|^2) \varphi(|X_1|^{2r})^{1/r} \varphi\left(\left|\sum_{j=1}^o U_j V_j\right|^{2r}\right)^{1/2r} \\
&\leq |t-s|^{2\gamma} 2^{-(1-2\gamma)n} \|X_1\|^4 \left\| \sum_{j=1}^o U_j V_j \right\| \\
&\leq |t-s|^{2\gamma} 2^{-(1-2\gamma)n} \|X_1\|^4 \left(\sum_{j=1}^o \|U_j\| \|V_j\| \right). \tag{27}
\end{aligned}$$

It is easy to see that the above arguments could also be applied to the more general situation where $\mathbf{U} := \sum_{j=1}^o U_j \otimes V_j$ with

$$U_j := \sum_{k=0}^{K_j} \alpha_{j,k} X_{s_1^{j,k}} \cdots X_{s_{m_j,k}^{j,k}} \quad \text{and} \quad V_j := \sum_{\ell=0}^{L_j} \beta_{j,\ell} X_{u_1^{j,\ell}} \cdots X_{u_{p_j,\ell}^{j,\ell}}, \quad \alpha_{j,k}, \beta_{j,\ell} \in \mathbb{C}, \quad s_a^{j,k}, u_b^{j,\ell} \in [0, s],$$

leading in the end to the same bound (27). Therefore, this bound (27) can actually be extended to any $U_j, V_j \in \mathcal{A}_s$, which then entails that for every $\mathbf{U} \in \mathcal{A}_s \hat{\otimes} \mathcal{A}_s$,

$$\varphi(|S_{st}^{1,n}[\mathbf{U}]|^{2r})^{1/2r} \leq |t-s|^{2\gamma} 2^{-(1-2\gamma)n} \|X_1\|^4 \|\mathbf{U}\|,$$

and by letting r tend to infinity, we get by (1) that

$$\|S_{st}^{1,n}[\mathbf{U}]\| \leq |t-s|^{2\gamma} 2^{-(1-2\gamma)n} \|X_1\|^4 \|\mathbf{U}\|.$$

The very same reasoning can of course be used in order to estimate $\|S_{st}^{2,n}[\mathbf{U}]\|$, with the same resulting bound. Going back to (21) and (22), we have thus proved that $\mathbb{X}^{2,n}$ is a Cauchy sequence in $\mathcal{C}_2^\lambda(\mathcal{L}_T(\mathcal{A}_\rightarrow))$, and by Lemma 3.3, we can therefore assert that it converges in this space to some element $\mathbb{X}^{2,S}$.

The product Chen identity (10) for $\mathbb{X}^{2,S}$ is readily obtained by passing to the limit (in a pointwise way) in the product Chen identity that is trivially satisfied by $\mathbb{X}^{2,n}$. Finally, in order to show that $\mathbb{X}^{2,S}$ actually belongs to $\mathcal{C}_2^\lambda(\mathcal{L}_T(\mathcal{A}_\rightarrow))$, fix $s < t$, $\mathbf{U} \in \mathcal{A}_s \hat{\otimes} \mathcal{A}_s$, and set

$$W^n := \mathbb{X}_{st}^{2,n}[\mathbf{U}], \quad W := \mathbb{X}_{st}^{2,S}[\mathbf{U}], \quad \bar{W}^n := \int_s^{t_\ell^n} (\mathbf{U} \sharp \delta X_{su}^n) dX_u^n,$$

where t_ℓ^n is such that $s < t_\ell^n \leq t < t_{\ell+1}^n$ (considering n large enough). Using the first estimate in (20), it is easy to check that $\|W^n - \bar{W}^n\| \rightarrow 0$, and so, since $\|W^n - W\| \rightarrow 0$, we get that $\|\bar{W}^n - W\| \rightarrow 0$. As $\bar{W}^n \in \mathcal{A}_t$, we can conclude that $W \in \mathcal{A}_t$, as expected. \square

Remark 3.4. Observe that in a commutative setting, the sum (22) would simply vanish, leading to an almost trivial proof, which clearly points out the specificity of our non-commutative framework (as evoked in Remark 2.3).

4. COMPARISON WITH $L^2(\varphi)$ -CONSTRUCTIONS

Our objective in this section is to compare the previous $L^\infty(\varphi)$ -constructions (i.e., constructions based on the operator norm) with the $L^2(\varphi)$ -constructions exhibited by Donati-Martin in [8]. In brief, we shall see that, when studied in $L^2(\varphi)$, the previous rough constructions correspond to Stratonovich-type integrals, while the constructions in [8] are more of a Itô-type. This comparison relies on an additional ingredient, the so-called second-quantization operator, whose central role in q -integration theory was already pointed out in Donati-Martin's work.

Since we intend to make specific references to some of the results of [8], we assume for simplicity that we are exactly in the same setting as in the latter study. Namely, for a fixed $q \in [0, 1)$, we assume that the q -Bm $\{X_t\}_{t \geq 0}$ we will handle in this section is constructed as the “canonical process” on the q -Fock space (\mathcal{A}, φ) (see [8] for details on these structures).

As in the previous sections, we denote by \mathcal{A}_t the closure, with respect to the operator norm, of the algebra generated by $\{X_s\}_{s \leq t}$.

4.1. Second quantization. Recall that the space $L^2(\varphi)$ is defined as the completion of \mathcal{A} as a Hilbert space through the product

$$\langle U, V \rangle := \varphi(UV^*) . \quad (28)$$

We will denote by $\|\cdot\|_{L^2(\varphi)}$ the associated norm, to be distinguished from the operator norm $\|\cdot\|$. For every $t \geq 0$, let \mathcal{B}_t be the von Neumann algebra generated by $\{X_s\}_{s \leq t}$ (observe in particular that $\mathcal{A}_t \subset \mathcal{B}_t \subset \mathcal{A}$) and denote by $\varphi(\cdot|\mathcal{B}_t)$ the conditional expectation with respect to \mathcal{B}_t . In other words, for every $U \in \mathcal{A}$, $\varphi(U|\mathcal{B}_t)$ stands for the orthogonal projection of U onto \mathcal{B}_t , with respect to the product (28): $Z = \varphi(U|\mathcal{B}_t)$ if and only if $Z \in \mathcal{B}_t$ and $\varphi(ZW^*) = \varphi(UW^*)$ for every $W \in \mathcal{B}_t$.

A possible way to introduce the second-quantization operator goes through the following invariance result:

Lemma 4.1. [8, Theorem 3.1]. *For all $s_0 < t_0$, $s_1 < t_1$, with $s_0 \leq s_1$, and $U \in \mathcal{A}_{s_0} \subset \mathcal{A}_{s_1}$, it holds that*

$$\frac{\varphi((\delta X)_{s_0 t_0} U (\delta X)_{s_0 t_0} | \mathcal{B}_{s_0})}{|t_0 - s_0|} = \frac{\varphi((\delta X)_{s_1 t_1} U (\delta X)_{s_1 t_1} | \mathcal{B}_{s_1})}{|t_1 - s_1|} .$$

Definition 4.2. *We call second quantization of X the operator $\Gamma_q : \cup_{t \geq 0} \mathcal{A}_t \rightarrow \mathcal{A}$ defined for all $s \geq 0$ and $U \in \mathcal{A}_s$ by the formula*

$$\Gamma_q(U) := \varphi((\delta X)_{s, s+1} U (\delta X)_{s, s+1} | \mathcal{B}_s) .$$

In particular, for all $s \geq 0$ and $U \in \mathcal{A}_s$, $\Gamma_q(U) \in \mathcal{B}_s$, $\Gamma_q(U)^ = \Gamma_q(U^*)$ and*

$$\|\Gamma_q(U)\|_{L^2(\varphi)} \leq \|(\delta X)_{s, s+1} U (\delta X)_{s, s+1}\|_{L^2(\varphi)} \leq \|X_1\|^2 \|U\| . \quad (29)$$

Remark 4.3. For $q = 0$, it is easy to check that, thanks to the freeness properties of X , the second quantization reduces to $\Gamma_0(U) = \varphi(U)$, while in the commutative situation, that is when $q \rightarrow 1$, one has (at least morally) $\Gamma_1(U) = U$.

In fact, we will essentially use the operator Γ_q through the following result, which offers a quite general tool to study Itô/Stratonovich correction terms (for the sake of clarity, we have postponed the proof of this proposition to Section 4.4):

Proposition 4.4. *For every adapted triprocess $\mathcal{U} \in \mathcal{C}_1^\varepsilon([s, t]; \mathcal{A}^{\otimes 3})$ ($s < t$, $\varepsilon > 0$) and every subdivision Δ of $[s, t]$ whose mesh $|\Delta|$ tends to 0, it holds that*

$$\sum_{(t_i) \in \Delta} (\delta X_{t_i t_{i+1}} \sharp \mathcal{U}_{t_i}) \sharp \delta X_{t_i t_{i+1}} \longrightarrow \int_s^t [Id \times \Gamma_q \times Id](\mathcal{U}_u) du \quad \text{in } L^2(\varphi) ,$$

where $Id \times \Gamma_q \times Id$ stands for the continuous extension, as an operator from $\cup_{u \geq 0} \mathcal{A}_u^{\otimes 3}$ to $L^2(\varphi)$, of the operator

$$(Id \times \Gamma_q \times Id)(U_1 \otimes U_2 \otimes U_3) := U_1 \Gamma_q(U_2) U_3 \quad , \quad U_1, U_2, U_3 \in \mathcal{A}_u .$$

4.2. Non-commutative Itô integral. Let us here slightly rephrase the results of [8] regarding Itô's approach to stochastic integration with respect to X .

Definition 4.5. *Fix an interval $I \subset \mathbb{R}$. An adapted biprocess $\mathbf{U} : I \rightarrow \mathcal{A} \hat{\otimes} \mathcal{A}$ is said to be Itô integrable against X if it is adapted and if for every partition Δ of I whose mesh $|\Delta|$ tends to 0, the sequence of Riemann sums*

$$S_X^\Delta(\mathbf{U}) := \sum_{t_i \in \Delta} \mathbf{U}_{t_i} \sharp \delta X_{t_i t_{i+1}}$$

converges in $L^2(\varphi)$ (as $|\Delta| \rightarrow 0$). In this case, we call the limit of $S_X^\Delta(\mathbf{U})$ the product Itô integral of \mathbf{U} against X , and we denote it by

$$\int_I \mathbf{U}_s \sharp dX_s \in L^2(\varphi) .$$

Given a biprocess $\mathbf{U} : I \rightarrow \mathcal{A} \hat{\otimes} \mathcal{A}$ and a partition Δ of I , we denote by \mathbf{U}^Δ the step-approximation

$$\mathbf{U}^\Delta := \sum_{t_i \in \Delta} \mathbf{U}_{t_i} \mathbf{1}_{[t_i, t_{i+1}[} .$$

The following isometry property, to be compared with the classical Brownian Itô isometry, is the key ingredient to identify Itô-integrable processes:

Proposition 4.6. [8, Proposition 3.3]. *For every interval $I \subset \mathbb{R}$, all adapted biprocesses*

$$\mathbf{U} : I \rightarrow \mathcal{A} \otimes \mathcal{A} \quad , \quad \mathbf{V} : I \rightarrow \mathcal{A} \otimes \mathcal{A} \quad ,$$

and all partitions Δ_1, Δ_2 of I , it holds that

$$\langle S_X^{\Delta_1}(\mathbf{U}), S_X^{\Delta_2}(\mathbf{V}) \rangle_{L^2(\varphi)} = \int_0^\infty \langle \langle \mathbf{U}_u^{\Delta_1}, \mathbf{V}_u^{\Delta_2} \rangle \rangle_q du \quad , \quad (30)$$

where $\langle \langle \cdot, \cdot \rangle \rangle_q$ is the bilinear extension of the application defined for all $U_1, U_2, V_1, V_2 \in \cup_{t \geq 0} \mathcal{A}_t$ as

$$\langle \langle U_1 \otimes U_2, V_1 \otimes V_2 \rangle \rangle_q := \varphi(U_1 \Gamma_q(U_2 V_2^*) V_1^*) \quad .$$

Corollary 4.7. *Let $\mathbf{U} : I \rightarrow \hat{\mathcal{A}} \hat{\otimes} \mathcal{A}$ be an adapted biprocess such that*

$$\int_I \|\mathbf{U}_u\|_{\hat{\mathcal{A}} \hat{\otimes} \mathcal{A}}^2 du < \infty \quad \text{and} \quad \int_I \|\mathbf{U}_u^\Delta - \mathbf{U}_u\|_{\hat{\mathcal{A}} \hat{\otimes} \mathcal{A}}^2 du \rightarrow 0 \quad \text{as } |\Delta| \rightarrow 0 \quad ,$$

for every partition Δ of I . Then \mathbf{U} is Itô integrable against X and

$$\left\| \int_I \mathbf{U}_u \# dX_u \right\|_{L^2(\varphi)}^2 = \int_I \langle \langle \mathbf{U}_u, \mathbf{U}_u \rangle \rangle_q du \quad . \quad (31)$$

Proof. Let us just provide a few details, the procedure being essential standard. Consider a sequence $\mathbf{U}^n : I \rightarrow \mathcal{A} \otimes \mathcal{A}$ of adapted biprocesses such that for every $t \in I$, $\|\mathbf{U}_t^n - \mathbf{U}_t\| \rightarrow 0$. Then, given two partitions Δ_1, Δ_2 of I , one has by (30)

$$\|S_X^{\Delta_1}(\mathbf{U}^n) - S_X^{\Delta_2}(\mathbf{U}^n)\|_{L^2(\varphi)}^2 = \int_I \langle \langle \mathbf{U}_u^{n, \Delta_1} - \mathbf{U}_u^{n, \Delta_2}, \mathbf{U}_u^{n, \Delta_1} - \mathbf{U}_u^{n, \Delta_2} \rangle \rangle_q du \quad .$$

By applying Cauchy-Schwarz inequality and then (29), it is readily checked that for all $\mathbf{V} \in \mathcal{A}_s \otimes \mathcal{A}_s$,

$$\langle \langle \mathbf{V}, \mathbf{V} \rangle \rangle_q \leq \|X_1\|^2 \|\mathbf{V}\|^2 \quad ,$$

and so

$$\|S_X^{\Delta_1}(\mathbf{U}^n) - S_X^{\Delta_2}(\mathbf{U}^n)\|_{L^2(\varphi)}^2 \leq \|X_1\|^2 \int_I \|\mathbf{U}_u^{n, \Delta_1} - \mathbf{U}_u^{n, \Delta_2}\|^2 du \quad ,$$

which, by letting n tend to infinity, leads us to

$$\|S_X^{\Delta_1}(\mathbf{U}) - S_X^{\Delta_2}(\mathbf{U})\|_{L^2(\varphi)}^2 \leq \|X_1\|^2 \int_I \|\mathbf{U}_u^{\Delta_1} - \mathbf{U}_u^{\Delta_2}\|^2 du \quad .$$

The conclusion easily follows. \square

4.3. Comparison with the rough integral. We now have all the tools to identify, as elements in $L^2(\varphi)$, the rough constructions arising from Sections 2 and 3.

Let us first consider the situation at the level of the product Lévy area provided by Theorem 3.1. To this end, given $0 \leq s < t$ and $\mathbf{U} \in \mathcal{A}_s \hat{\otimes} \mathcal{A}_s$, observe that, by Corollary 4.7, the biprocess $\mathbf{V}_u := (\mathbf{U} \# \delta X_{su}) \otimes 1$ is known to be Itô-integrable on $[s, t]$, which allows us to consider the integral

$$\int_s^t (\mathbf{U} \# \delta X_{su}) dX_u \in L^2(\varphi) \quad .$$

Proposition 4.8. *For all $0 \leq s < t$ and every $\mathbf{U} \in \mathcal{A}_s \hat{\otimes} \mathcal{A}_s$, it holds that*

$$\mathbb{X}_{st}^{2, S}[\mathbf{U}] = \int_s^t (\mathbf{U} \# \delta X_{su}) dX_u + \frac{1}{2}(t-s)(Id \times \Gamma_q)[\mathbf{U}] \quad \text{in } L^2(\varphi) \quad , \quad (32)$$

where $Id \times \Gamma_q$ stands for the continuous extension, as an operator from $\mathcal{A}_s \hat{\otimes} \mathcal{A}_s$ to $L^2(\varphi)$, of the operator

$$(Id \times \Gamma_q)[U \otimes V] := U \Gamma_q(V) \quad .$$

Proof. Fix $s < t$, $\mathbf{U} \in \mathcal{A}_s \hat{\otimes} \mathcal{A}_s$, and let \tilde{D}^n be the subdivision obtained by adding the two times s, t to the dyadic partition $D^n := \{i/2^n, i \geq 0\}$. Denote by \tilde{X}^n the linear interpolation of X along \tilde{D}^n and set $\hat{X}^n := \sum_{\hat{t}_i} X_{\hat{t}_i} 1_{[\hat{t}_i, \hat{t}_{i+1})}$ where $\{s = \hat{t}_1 < \dots < \hat{t}_n = t\} := \tilde{D}^n \cap [s, t]$. Besides, we recall that the notation \mathbb{X}^{2, D^n} (or $\mathbb{X}^{2, \tilde{D}^n}$) has been introduced in (14).

Using only the 1/2-Hölder regularity of X (see (5)), it is easy to check that for every $\mathbf{U} \in \mathcal{A}_s \hat{\otimes} \mathcal{A}_s$,

$$\|\mathbb{X}_{st}^{2, D^n}[\mathbf{U}] - \mathbb{X}_{st}^{2, \tilde{D}^n}[\mathbf{U}]\| \leq c \|X_1\|^2 \|\mathbf{U}\| |t-s|^{2\gamma} 2^{-n(1/2-\gamma)} \quad , \quad (33)$$

for some universal constant c and for every $\gamma \in (0, 1/2)$. Thus, by Theorem 3.1, we can assert that $\mathbb{X}_{st}^{2, \tilde{D}^n}[\mathbf{U}]$ converges to $\mathbb{X}_{st}^{2, S}[\mathbf{U}]$ for the operator norm (and accordingly in $L^2(\varphi)$). Now write

$$\mathbb{X}_{st}^{2, \tilde{D}^n}[\mathbf{U}] = \int_s^t (\mathbf{U} \sharp \delta \tilde{X}_{su}^n) d\tilde{X}_u^n \quad (34)$$

$$\begin{aligned} &= \sum_{k=1}^{n-1} \frac{1}{\hat{t}_{k+1} - \hat{t}_k} \int_{\hat{t}_k}^{\hat{t}_{k+1}} \mathbf{U} \sharp \left(\delta X_{s\hat{t}_k} + \frac{u - \hat{t}_k}{\hat{t}_{k+1} - \hat{t}_k} (\delta X)_{\hat{t}_k \hat{t}_{k+1}} \right) du (\delta X)_{\hat{t}_k \hat{t}_{k+1}} \\ &= \sum_{k=1}^{n-1} (\mathbf{U} \sharp \delta X_{s\hat{t}_k}) \delta X_{\hat{t}_k \hat{t}_{k+1}} + \frac{1}{2} \sum_{k=1}^{n-1} \mathbf{U} \sharp (\delta X)_{\hat{t}_k \hat{t}_{k+1}} (\delta X)_{\hat{t}_k \hat{t}_{k+1}} \\ &= \int_s^t (\mathbf{U} \sharp \delta \hat{X}_{su}^n) dX_u + \frac{1}{2} \sum_{k=1}^{n-1} \mathbf{U} \sharp (\delta X)_{\hat{t}_k \hat{t}_{k+1}} (\delta X)_{\hat{t}_k \hat{t}_{k+1}}. \end{aligned} \quad (35)$$

Thanks to (31), it holds that

$$\begin{aligned} &\left\| \int_s^t (\mathbf{U} \sharp \delta \hat{X}_{su}^n) dX_u - \int_s^t (\mathbf{U} \sharp \delta X_{su}) dX_u \right\|_{L^2(\varphi)}^2 \\ &= \int_s^t \langle (\mathbf{U} \sharp [\delta \hat{X}_{su}^n - \delta X_{su}]) \otimes 1, (\mathbf{U} \sharp [\delta \hat{X}_{su}^n - \delta X_{su}]) \otimes 1 \rangle_q du \\ &= \int_s^t \|\mathbf{U} \sharp [\delta \hat{X}_{su}^n - \delta X_{su}]\|_{L^2(\varphi)}^2 du \\ &\leq \|\mathbf{U}\|^2 \sum_{k=1}^{n-1} \int_{\hat{t}_k}^{\hat{t}_{k+1}} \|X_{\hat{t}_k} - X_u\|^2 du \\ &\leq \|\mathbf{U}\|^2 \sum_{k=1}^{n-1} \int_{\hat{t}_k}^{\hat{t}_{k+1}} (u - \hat{t}_k) du \leq \frac{1}{2} \|\mathbf{U}\|^2 2^{-n} |t - s| \rightarrow 0. \end{aligned}$$

Observe finally that the limit of the second term in (35) is immediately provided by Proposition 4.4, which achieves the proof of (32). \square

Let us now extend the correction formula (32) to any adapted controlled biprocess, that is to the class of biprocesses introduced in Definition 2.4. Using again Corollary 4.7, it is easy to check that, as an adapted Hölder path in $\mathcal{A} \hat{\otimes} \mathcal{A}$, any such controlled biprocess is Itô-integrable when considered on an interval I of finite Lebesgue measure. This puts us in a position to state the formula:

Corollary 4.9. *For all $0 \leq s < t$ and every adapted controlled biprocess $\mathbf{U} \in \mathbf{Q}_X([s, t])$ with decomposition (12), it holds that*

$$\int_s^t \mathbf{U}_u \sharp d\mathbb{X}_u^S = \int_s^t \mathbf{U}_u \sharp dX_u + \frac{1}{2} \int_s^t (Id \times \Gamma_q \times Id) [\mathcal{U}_u^{X,1} + \mathcal{U}_u^{X,2}] du \quad \text{in } L^2(\varphi). \quad (36)$$

Proof. The transition from (32) to (36) follows from the very same Taylor-expansion argument as in the proof of [7, Proposition 5.6] (related to the free case), and so, for the sake of conciseness, we do not repeat it here. \square

At this point, observe that the combination of Proposition 2.9 and Corollary 4.9 immediately yields the following q -extension of Itô/Stratonovich formula: for all $f \in \mathbf{F}_3$ and $s < t$,

$$\delta(f(X))_{st} = \int_s^t \partial f(X_u) \sharp d\mathbb{X}_u^S = \int_s^t \partial f(X_u) \sharp dX_u + \int_s^t [Id \times \Gamma_q \times Id] (\partial^2 f(X_u)) du.$$

As another spin-off of Formula (36), we can finally derive an expression of the rough Stratonovich integral $\int_s^t \mathbf{U}_u \sharp d\mathbb{X}_u^S$ as the $L^2(\varphi)$ -limit of “mean-value” Riemann sums. The result, which emphasizes the analogy between the rough construction and the classical (commutative) Stratonovich integral, can be stated as follows:

Corollary 4.10. *For all $0 \leq s < t$ and every adapted controlled biprocess $\mathbf{U} \in \mathbf{Q}_X([s, t])$, it holds that*

$$\int_s^t \mathbf{U}_u \# d\mathbb{X}_u^S = \lim_{|\Delta| \rightarrow 0} \sum_{(t_i) \in \Delta} \frac{1}{2} (\mathbf{U}_{t_i} + \mathbf{U}_{t_{i+1}}) \# \delta X_{t_i t_{i+1}} \quad \text{in } L^2(\varphi) , \quad (37)$$

for any subdivision Δ of $[s, t]$ whose mesh $|\Delta|$ tends to 0.

Proof. For any subdivision $\Delta = (t_i)$ of $[s, t]$, write

$$\begin{aligned} & \frac{1}{2} (\mathbf{U}_{t_i} + \mathbf{U}_{t_{i+1}}) \# \delta X_{t_i t_{i+1}} \\ &= \mathbf{U}_{t_i} \# \delta X_{t_i t_{i+1}} + \frac{1}{2} \delta \mathbf{U}_{t_i t_{i+1}} \# \delta X_{t_i t_{i+1}} \\ &= \mathbf{U}_{t_i} \# \delta X_{t_i t_{i+1}} + \frac{1}{2} [(\delta X_{t_i t_{i+1}} \# \mathcal{U}_{t_i}^{X,1}) \# \delta X_{t_i t_{i+1}} + (\mathcal{U}_{t_i}^{X,2} \# \delta X_{t_i t_{i+1}}) \# \delta X_{t_i t_{i+1}} + \mathbf{U}_{t_i t_{i+1}}^b \# \delta X_{t_i t_{i+1}}] , \end{aligned}$$

and observe that, with the notations of Section 2.3, we have

$$\|\mathbf{U}_{t_i t_{i+1}}^b \# \delta X_{t_i t_{i+1}}\| \leq |t_{i+1} - t_i|^{2\gamma+1/2} \|X_1\| \mathcal{N}[\mathbf{U}^b; \mathcal{C}_2^{2\gamma}([s, t])] .$$

Taking the sum over i and then letting $|\Delta|$ tend to 0, we get by Proposition 4.4 that the sum in (37) converges in $L^2(\varphi)$ to the right-hand side of (36), which leads us to the conclusion. \square

4.4. Proof of Proposition 4.4. It is based on similar conditioning arguments as in the proof of [8, Theorem 3.2]. Let $\mathcal{U}^n : [s, t] \rightarrow \mathcal{A}^{\otimes 3}$ be a sequence of adapted triprocesses such that $\|\mathcal{U}_u^n - \mathcal{U}_u\| \rightarrow 0$ for every $u \in [s, t]$, and fix a subdivision $\Delta = (t_i)$ of $[s, t]$. Then set successively $Y_i := \delta X_{t_i t_{i+1}}$,

$$S_\Delta(\mathcal{U}) := \sum_{(t_i) \in \Delta} \{ (Y_i \# \mathcal{U}_{t_i}) \# Y_i - (t_{i+1} - t_i) [\text{Id} \times \Gamma_q \times \text{Id}] (\mathcal{U}_{t_i}) \}$$

$$\text{and } S_\Delta^n(\mathcal{U}) := \sum_{(t_i) \in \Delta} \{ (Y_i \# \mathcal{U}_{t_i}^n) \# Y_i - (t_{i+1} - t_i) [\text{Id} \times \Gamma_q \times \text{Id}] (\mathcal{U}_{t_i}^n) \} .$$

If $\mathcal{U}_t^n := \sum_{\ell \leq L_t^n} U_{t,\ell}^n \otimes V_{t,\ell}^n \otimes W_{t,\ell}^n \in \mathcal{A}^{\otimes 3}$, $S_\Delta^n(\mathcal{U})$ thus corresponds to $S_\Delta^n(\mathcal{U}) = \sum_{(t_i) \in \Delta} \sum_{\ell \leq L_{t_i}^n} M_{i,\ell}^n$, with

$$M_{i,\ell}^n := U_{t_i,\ell}^n Y_i V_{t_i,\ell}^n Y_i W_{t_i,\ell}^n - (t_{i+1} - t_i) U_{t_i,\ell}^n \Gamma_q(V_{t_i,\ell}^n) W_{t_i,\ell}^n ,$$

so

$$\|S_\Delta^n(\mathcal{U})\|_{L^2(\varphi)}^2 = \sum_{(t_{i_1}) \in \Delta} \sum_{(t_{i_2}) \in \Delta} \sum_{\ell_1 \leq L_{t_{i_1}}^n} \sum_{\ell_2 \leq L_{t_{i_2}}^n} \varphi(M_{i_1,\ell_1}^n (M_{i_2,\ell_2}^n)^*) . \quad (38)$$

For more clarity, let us set $U_{i,\ell}^n := U_{t_i,\ell}^n$, $V_{i,\ell}^n := V_{t_i,\ell}^n$, $W_{i,\ell}^n := W_{t_i,\ell}^n$, and consider then the expansion

$$\begin{aligned} & \varphi(M_{i_1,\ell_1}^n (M_{i_2,\ell_2}^n)^*) \\ &= \varphi(U_{i_1,\ell_1}^n Y_{i_1} V_{i_1,\ell_1}^n Y_{i_1} W_{i_1,\ell_1}^n W_{i_2,\ell_2}^{n,*} Y_{i_2} V_{i_2,\ell_2}^{n,*} Y_{i_2} U_{i_2,\ell_2}^{n,*}) \\ &\quad - (t_{i_2+1} - t_{i_2}) \varphi(U_{i_1,\ell_1}^n Y_{i_1} V_{i_1,\ell_1}^n Y_{i_1} W_{i_1,\ell_1}^n W_{i_2,\ell_2}^{n,*} \Gamma_q(V_{i_2,\ell_2}^{n,*}) U_{i_2,\ell_2}^{n,*}) \\ &\quad - (t_{i_1+1} - t_{i_1}) \varphi(U_{i_1,\ell_1}^n \Gamma_q(V_{i_1,\ell_1}^n) W_{i_1,\ell_1}^n W_{i_2,\ell_2}^{n,*} Y_{i_2} V_{i_2,\ell_2}^{n,*} Y_{i_2} U_{i_2,\ell_2}^{n,*}) \\ &\quad + (t_{i_1+1} - t_{i_1})(t_{i_2+1} - t_{i_2}) \varphi(U_{i_1,\ell_1}^n \Gamma_q(V_{i_1,\ell_1}^n) W_{i_1,\ell_1}^n W_{i_2,\ell_2}^{n,*} \Gamma_q(V_{i_2,\ell_2}^{n,*}) U_{i_2,\ell_2}^{n,*}) . \end{aligned} \quad (39)$$

Step 1: Non-diagonal terms ($i_1 \neq i_2$). Observe first that if for instance $i_1 < i_2$, we have, by combining Lemma 4.1 and Definition 4.2,

$$\begin{aligned} & \varphi(U_{i_1,\ell_1}^n Y_{i_1} V_{i_1,\ell_1}^n Y_{i_1} W_{i_1,\ell_1}^n W_{i_2,\ell_2}^{n,*} Y_{i_2} V_{i_2,\ell_2}^{n,*} Y_{i_2} U_{i_2,\ell_2}^{n,*}) \\ &= \varphi(U_{i_1,\ell_1}^n Y_{i_1} V_{i_1,\ell_1}^n Y_{i_1} W_{i_1,\ell_1}^n W_{i_2,\ell_2}^{n,*} \varphi(Y_{i_2} V_{i_2,\ell_2}^{n,*} Y_{i_2} | \mathcal{B}_{t_{i_2}}) U_{i_2,\ell_2}^{n,*}) \\ &= (t_{i_2+1} - t_{i_2}) \varphi(U_{i_1,\ell_1}^n Y_{i_1} V_{i_1,\ell_1}^n Y_{i_1} W_{i_1,\ell_1}^n W_{i_2,\ell_2}^{n,*} \Gamma_q(V_{i_2,\ell_2}^{n,*}) U_{i_2,\ell_2}^{n,*}) , \end{aligned}$$

and with the same conditioning argument

$$\begin{aligned} & \varphi(U_{i_1, \ell_1}^n \Gamma_q(V_{i_1, \ell_1}^n) W_{i_1, \ell_1}^n W_{i_2, \ell_2}^{n,*} Y_{i_2} V_{i_2, \ell_2}^{n,*} Y_{i_2} U_{i_2, \ell_2}^{n,*}) \\ &= (t_{i_2+1} - t_{i_2}) \varphi(U_{i_1, \ell_1}^n \Gamma_q(V_{i_1, \ell_1}^n) W_{i_1, \ell_1}^n W_{i_2, \ell_2}^{n,*} \Gamma_q(V_{i_2, \ell_2}^{n,*}) U_{i_2, \ell_2}^{n,*}) , \end{aligned}$$

so that, going back to (39), one has $\varphi(M_{i_1, \ell_1}^n (M_{i_2, \ell_2}^n)^*) = 0$. Similar arguments lead to the same conclusion when $i_2 < i_1$.

Step 2: Diagonal terms ($i_1 = i_2 = i$). First, observe that with the same conditioning argument as above, decomposition (39) actually reduces to

$$\begin{aligned} \varphi(M_{i, \ell_1}^n (M_{i, \ell_2}^n)^*) &= \varphi(U_{i, \ell_1}^n Y_i V_{i, \ell_1}^n Y_i W_{i, \ell_1}^n W_{i, \ell_2}^{n,*} Y_i V_{i, \ell_2}^{n,*} Y_i U_{i, \ell_2}^{n,*}) \\ &\quad - (t_{i+1} - t_i)^2 \varphi(U_{i, \ell_1}^n \Gamma_q(V_{i, \ell_1}^n) W_{i, \ell_1}^n W_{i, \ell_2}^{n,*} \Gamma_q(V_{i, \ell_2}^{n,*}) U_{i, \ell_2}^{n,*}) . \end{aligned}$$

Now, on the one hand, using (5) and the Cauchy-Schwarz inequality,

$$\begin{aligned} & |\varphi(U_{i, \ell_1}^n Y_i V_{i, \ell_1}^n Y_i W_{i, \ell_1}^n W_{i, \ell_2}^{n,*} Y_i V_{i, \ell_2}^{n,*} Y_i U_{i, \ell_2}^{n,*})| \\ &\leq \|Y_i\|^4 \|U_{i, \ell_1}^n\| \|V_{i, \ell_1}^n\| \|W_{i, \ell_1}^n\| \|W_{i, \ell_2}^{n,*}\| \|V_{i, \ell_2}^{n,*}\| \|U_{i, \ell_2}^n\| \\ &\leq (t_{i+1} - t_i)^2 \|X_1\|^4 \|U_{i, \ell_1}^n\| \|V_{i, \ell_1}^n\| \|W_{i, \ell_1}^n\| \|W_{i, \ell_2}^{n,*}\| \|V_{i, \ell_2}^{n,*}\| \|U_{i, \ell_2}^n\| . \end{aligned}$$

On the other hand, using the definition of $\Gamma_q(V_{j_1})$,

$$\begin{aligned} & |\varphi(U_{i, \ell_1}^n \Gamma_q(V_{i, \ell_1}^n) W_{i, \ell_1}^n W_{i, \ell_2}^{n,*} \Gamma_q(V_{i, \ell_2}^{n,*}) U_{i, \ell_2}^{n,*})| \\ &= |\varphi(U_{i, \ell_1}^n (\delta X)_{t_i, t_{i+1}} V_{i, \ell_1}^n (\delta X)_{t_i, t_{i+1}} W_{i, \ell_1}^n W_{i, \ell_2}^{n,*} \Gamma_q(V_{i, \ell_2}^{n,*}) U_{i, \ell_2}^{n,*})| \\ &\leq \|U_{i, \ell_2}^{n,*} U_{i, \ell_1}^n (\delta X)_{t_i, t_{i+1}} V_{i, \ell_1}^n (\delta X)_{t_i, t_{i+1}} W_{i, \ell_1}^n W_{i, \ell_2}^{n,*}\|_{L^2(\varphi)} \|\Gamma_q(V_{i, \ell_2}^{n,*})\|_{L^2(\varphi)} , \end{aligned}$$

which, by (29), entails that

$$|\varphi(U_{i, \ell_1}^n \Gamma_q(V_{i, \ell_1}^n) W_{i, \ell_1}^n W_{i, \ell_2}^{n,*} \Gamma_q(V_{i, \ell_2}^{n,*}) U_{i, \ell_2}^{n,*})| \leq \|X_1\|^4 \|U_{i, \ell_1}^n\| \|V_{i, \ell_1}^n\| \|W_{i, \ell_1}^n\| \|W_{i, \ell_2}^{n,*}\| \|V_{i, \ell_2}^{n,*}\| \|U_{i, \ell_2}^n\| .$$

Going back to (38), we have thus shown that

$$\|S_\Delta^n(\mathcal{U})\|_{L^2(\varphi)}^2 \leq \|X_1\|^4 \sum_{(t_i) \in \Delta} (t_{i+1} - t_i)^2 \left(\sum_{\ell \leq L_{t_i}^n} \|U_{i, \ell}^n\| \|V_{i, \ell}^n\| \|W_{i, \ell}^n\| \right)^2 ,$$

and so we can assert that

$$\begin{aligned} \|S_\Delta^n(\mathcal{U})\|_{L^2(\varphi)}^2 &\leq \|X_1\|^4 \sum_{(t_i) \in \Delta} (t_{i+1} - t_i)^2 \|\mathcal{U}_{t_i}^n\|^2 \\ &\leq 2 \|X_1\|^4 \left\{ \sum_{(t_i) \in \Delta} (t_{i+1} - t_i)^2 \|\mathcal{U}_{t_i}^n - \mathcal{U}_{t_i}\|^2 + \left(\sup_{u \in [s, t]} \|\mathcal{U}_u\|^2 \right) |t - s| |\Delta| \right\} . \end{aligned}$$

By letting n tend to infinity first, we can conclude that $\|S_\Delta(\mathcal{U})\|_{L^2(\varphi)}^2 \rightarrow 0$ as $|\Delta| \rightarrow 0$. The convergence

$$\sum_{(t_i) \in \Delta} (t_{i+1} - t_i) [\text{Id} \times \Gamma_q \times \text{Id}](\mathcal{U}_{t_i}) \rightarrow \int_s^t [\text{Id} \times \Gamma_q \times \text{Id}](\mathcal{U}_u) du \quad \text{in } L^2(\varphi)$$

follows easily from the regularity of \mathcal{U} , by noting that for every u and every $\mathcal{V} \in \mathcal{A}_u^{\hat{\otimes} 3}$,

$$\|[\text{Id} \times \Gamma_q \times \text{Id}](\mathcal{V})\|_{L^2(\varphi)} \leq \|\mathcal{V}\| .$$

This achieves the proof of our statement.

5. GENERALITIES ON q -WIENER CHAOS

We now turn to the consideration of our second main objective in this q -Brownian analysis, namely a combinatorial description of the multiplication properties occurring in q -Wiener chaoses. As a first step, let us introduce in this section an alternative and as-graphical-as-possible approach to the notion of q -contraction, in comparison with the definition provided in [8].

5.1. Arrangement along a partition and q -contraction. We have seen in Section 1 that the law of a q -Gaussian family or a q -Bm can be conveniently described as a sum over pairings. In order to extend such formulas at the level of the processes (and not only their laws), we will need to involve more general “incomplete pairings” into the procedure:

Definition 5.1. (i) For $n \geq 1$, we denote by $\mathcal{P}_{\leq 2}(\{1, \dots, n\})$ or $\mathcal{P}_{\leq 2}(n)$ the set of partitions π of $\{1, \dots, n\}$ with blocks of one or two elements, and for every $0 \leq k \leq \frac{n}{2}$, we denote $\mathcal{P}_{\leq 2}^k$ the set of partitions $\pi \in \mathcal{P}_{\leq 2}$ containing exactly k pairs.

(ii) For $n_1, \dots, n_r \geq 1$, we denote by $\mathcal{P}_{\leq 2}(n_1 \otimes \dots \otimes n_r)$ the set of partitions $\pi \in \mathcal{P}_{\leq 2}(n_1 + \dots + n_r)$ respecting $n_1 \otimes \dots \otimes n_r$, i.e., with no pair contained in a same block $\{1, \dots, n_1\}, \{n_1 + 1, \dots, n_1 + n_2\}, \dots$. We define $\mathcal{P}_2(n_1 \otimes \dots \otimes n_r)$ and $\mathcal{P}_{\leq 2}^k(n_1 \otimes \dots \otimes n_r)$ along the same lines.

(iii) Given $\pi \in \mathcal{P}_{\leq 2}(n)$, a crossing in π is any set of the form $\{(a_1, b_1), (a_2, b_2)\}$ with $(a_i, b_i) \in \pi$ and $a_1 < a_2 < b_1 < b_2$, or of the form $\{(a, b), (c)\}$ with $(a, b) \in \pi$, $(c) \in \pi$ and $a < c < b$. We will denote by $\text{Cr}(\pi)$ the number of crossings in π .

Now observe that any partition $\pi \in \mathcal{P}_{\leq 2}^k(n)$ can be described in a unique way as

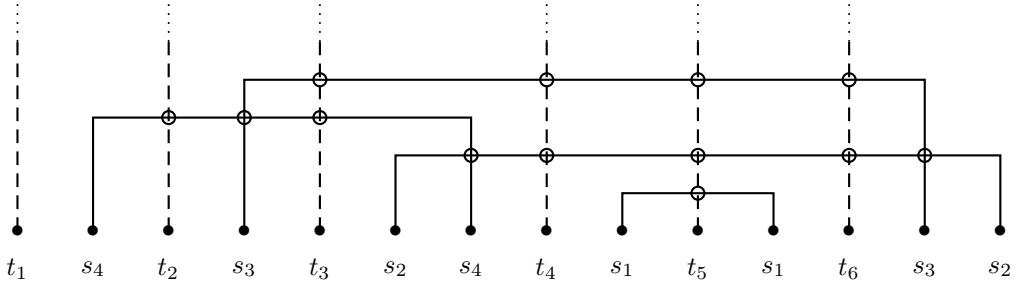
$$\pi = \{(a_i, b_i), i = 1, \dots, k, a_k < a_{k-1} < \dots < a_1\} \cup \{(c_i), i = 1, \dots, n - 2k, c_1 < c_2 < \dots < c_{n-2k}\}. \quad (40)$$

With this notation in mind, and for all $\mathbf{s} = (s_1, \dots, s_k) \in \mathbb{R}_+^k, \mathbf{t} = (t_1, \dots, t_{n-2k}) \in \mathbb{R}_+^{n-2k}$, we define the arrangement $\pi(\mathbf{s}, \mathbf{t}) \in \{\mathbf{s}, \mathbf{t}\}^n$ as follows: for $l = 1, \dots, n$,

$$\pi(\mathbf{s}, \mathbf{t})_l := \begin{cases} s_i & \text{if } l = a_i \text{ or } l = b_i, \\ t_i & \text{if } l = c_i. \end{cases}$$

Example 5.2. Consider the partition $\pi \in \mathcal{P}_{\leq 2}^4(14)$ drawn below, with singletons represented by dashed lines “extending to infinity” and pairs by continuous lines. Such a representation implies in particular that each pair $(a_i, b_i) \in \pi$ satisfying $a_i < c_j < b_i$, for some singleton $(c_j) \in \pi$, necessarily crosses the “line” (c_j) . It is then easy to visualize that in this case

$$\pi(\mathbf{s}, \mathbf{t}) = (t_1, s_4, t_2, s_3, t_3, s_2, s_4, t_4, s_1, t_5, s_1, t_6, s_3, s_2) \quad \text{and} \quad \text{Cr}(\pi) = 13.$$



The suitable definition of a contraction in our setting now reads as follows:

Definition 5.3. (i) Given $f \in L^2(\mathbb{R}_+^n)$ and $\pi \in \mathcal{P}_{\leq 2}^k(n)$, we call integral of f along π , and denote by $\int_\pi f$, the function defined for every $\mathbf{t} \in \mathbb{R}_+^{n-2k}$ by

$$\left[\int_\pi f \right](\mathbf{t}) := \int_{\mathbb{R}_+^k} d\mathbf{s} f(\pi(\mathbf{s}, \mathbf{t})),$$

where $\pi(\mathbf{s}, \mathbf{t})$ stands for the above-defined arrangement along π .

(ii) For all $f \in L^2(\mathbb{R}_+^m), g \in L^2(\mathbb{R}_+^n)$ and $\pi \in \mathcal{P}_{\leq 2}(m \otimes n)$, we define the contraction of f and g along π by the formula $f \otimes_\pi g := \int_\pi f \otimes g$. Finally, for all $q \in [0, 1)$ and $k \in \{0, \dots, m \wedge n\}$, we define the q -contraction of f and g of order k as the sum

$$f \otimes_k^q g := \sum_{\pi \in \mathcal{P}_{\leq 2}^k(m \otimes n)} q^{\text{Cr}(\pi)} f \otimes_\pi g.$$

This definition can be seen as a natural q -extension of the standard contracting procedure along a Feynman diagram (as defined in [12, Section 7.2]). It actually coincides with the notion of a q -contraction introduced in [8], which can be recovered by comparing [8, Proposition 4.1] with the subsequent Proposition 6.4 and applying the isometry property (43).

Notation 5.4. Given $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}_+^n$, we will denote by \mathbf{t}^* the vector (t_n, \dots, t_1) . We recall that for every $f \in L^2(\mathbb{R}_+^n)$, the mirror-symmetric of f , denoted by f^* , is the function in $L^2(\mathbb{R}_+^n)$ defined by $f^*(\mathbf{t}) = f(\mathbf{t}^*)$. Besides, given $\pi^* \in \mathcal{P}_{\leq 2}^k(m \otimes n)$, we define the mirror-symmetrized π^* of π as the element of $\mathcal{P}_{\leq 2}^k(n \otimes m)$ obtained by taking the symmetric of π along a vertical axis between the two blocks $\{1, \dots, m\}$ and $\{m+1, \dots, m+n\}$. Otherwise stated, if π is given by (40), then

$$\pi^* := \{(m+n+1-b_i, m+n+1-a_i), (m+n+1-c_j)\}.$$

For further use, let us also label the following readily-checked relations between mirror-symmetry and q -contractions:

Lemma 5.5. For all $f \in L^2(\mathbb{R}_+^m)$, $g \in L^2(\mathbb{R}_+^n)$ and $\pi \in \mathcal{P}_{\leq 2}(m \otimes n)$, it holds that

$$(f \otimes_\pi g)^* = g^* \otimes_{\pi^*} f^*.$$

Besides, $Cr(\pi^*) = Cr(\pi)$ and accordingly $(f \otimes_k^q g)^* = g^* \otimes_k^q f^*$ for every $k \in \{0, \dots, m \wedge n\}$.

5.2. Construction of the q -Wiener chaos. With the above “graphical” approach in mind, and also for the sake of completeness, let us briefly go back here to the definition of the q -Wiener chaos.

From now on and for the rest of the paper, we fix $q \in [0, 1)$ and consider a q -Bm $(X_t)_{t \geq 0} = (X^{(q)})_{t \geq 0}$ in some non-commutative probability space (\mathcal{A}, φ) . With the notations of Section 5.1, observe that the joint moments of $(X_t)_{t \geq 0}$ can also be expressed as

$$\varphi(X_{t_1} \dots X_{t_n}) = \sum_{\pi \in \mathcal{P}_2(n)} q^{Cr(\pi)} \int_\pi (\mathbf{1}_{[0, t_1]} \otimes \dots \otimes \mathbf{1}_{[0, t_n]}). \quad (41)$$

The construction of the multiple integrals of X , which give birth to the q -Wiener chaoses, can then be made along a similar procedure as in the classical commutative case (see [15]). Consider first the set \mathcal{E}_m of simple functions, that is the set of functions f of the form $f = \sum_{i_1, \dots, i_m=1}^p \lambda_{i_1, \dots, i_m} \mathbf{1}_{A_{i_1}} \otimes \dots \otimes \mathbf{1}_{A_{i_m}}$, for pairwise disjoint intervals A_i and coefficients $\lambda_{i_1, \dots, i_m}$ vanishing on diagonals. For such a function $f \in \mathcal{E}_m$, we set naturally

$$I_m^q(f) := \sum_{i_1, \dots, i_m=1}^p \lambda_{i_1, \dots, i_m} X(\mathbf{1}_{A_{i_1}}) \dots X(\mathbf{1}_{A_{i_m}}), \quad (42)$$

using convention that for any interval $A := [\ell_1, \ell_2]$, $X(\mathbf{1}_A) := X_{\ell_2} - X_{\ell_1}$. With this definition, observe in particular that $I_m^q(f)^* = I_m^q(f^*)$ for every $f \in \mathcal{E}_m$. Then, just as in the commutative case, the extension of $I_m^q(f)$ to any real-valued function $f \in L^2(\mathbb{R}_+^m)$ relies on two ingredients: the density of \mathcal{E}_m within $L^2(\mathbb{R}_+^m)$ on the one hand, an isometry property on the other, which, in this setting, reads as follows:

Proposition 5.6. For $f \in \mathcal{E}_m$ and $g \in \mathcal{E}_n$, it holds that

$$\langle I_m^q(f), I_n^q(g) \rangle_{L^2(\varphi)} = \delta_{m,n} \langle f, P_q(g) \rangle_{L^2(\mathbb{R}_+^n)}, \quad (43)$$

where P_q is the q -symmetrization operator defined for all $f \in L^2(\mathbb{R}_+^m)$ and $\mathbf{s} \in \mathbb{R}_+^m$ by

$$P_q(f)(\mathbf{s}) := \sum_{\pi \in \mathcal{P}_2(m \otimes m)} q^{Cr(\pi)} f(\pi(\mathbf{s})_{m+1::2m}) \quad , \quad \pi(\mathbf{s})_{m+1::2m} := (\pi(\mathbf{s})_{m+1}, \dots, \pi(\mathbf{s})_{2m}).$$

In particular, $\|P_q(f)\|_{L^2(\mathbb{R}_+^m)} \leq m! \|f\|_{L^2(\mathbb{R}_+^m)}$.

The resulting extension of the integral to any $f \in L^2(\mathbb{R}_+^m)$ will still be denoted by $I_m^q(f)$, as usual. It satisfies the isometry property (43), as well as $\varphi(I_m^q(f)) = 0$ for $m \geq 1$.

Proof of Proposition 5.6. Consider the case where

$$f := \sum_{i_1, \dots, i_m=1}^p \lambda_{i_1 \dots i_m} \mathbf{1}_{A_{i_1}} \otimes \dots \otimes \mathbf{1}_{A_{i_m}} \quad \text{and} \quad g := \sum_{i_1, \dots, i_n=1}^p \beta_{i_1 \dots i_n} \mathbf{1}_{A_{i_1}} \otimes \dots \otimes \mathbf{1}_{A_{i_n}},$$

for disjoint intervals (A_i) and coefficients λ, β vanishing on diagonals. Then by (41)

$$\begin{aligned} \varphi(I_m^q(f)^* I_n^q(g)) &= \sum_{i_1, \dots, i_m=1}^p \sum_{j_1, \dots, j_n=1}^p \lambda_{i_1 \dots i_m} \beta_{j_1 \dots j_n} \\ &\quad \sum_{\pi \in \mathcal{P}_2(m+n)} q^{\text{Cr}(\pi)} \int_{\pi} (\mathbf{1}_{A_{i_m}} \otimes \dots \otimes \mathbf{1}_{A_{i_1}} \otimes \mathbf{1}_{A_{j_1}} \otimes \dots \otimes \mathbf{1}_{A_{j_n}}) . \end{aligned} \quad (44)$$

Now, for pairwise distinct $i_1, \dots, i_m \in \{1, \dots, p\}$ and pairwise distinct $j_1, \dots, j_n \in \{1, \dots, p\}$, the integral

$$\int_{\pi} (\mathbf{1}_{A_{i_m}} \otimes \dots \otimes \mathbf{1}_{A_{i_1}} \otimes \mathbf{1}_{A_{j_1}} \otimes \dots \otimes \mathbf{1}_{A_{j_n}})$$

clearly vanishes for every $\pi \in \mathcal{P}_2(m+n)$ if $n \neq m$, and for every $\pi \in \mathcal{P}_2(2n) \setminus \mathcal{P}_2(n \otimes n)$ if $n = m$. Thus,

$$\begin{aligned} &\sum_{\pi \in \mathcal{P}_2(m+n)} q^{\text{Cr}(\pi)} \int_{\pi} (\mathbf{1}_{A_{i_m}} \otimes \dots \otimes \mathbf{1}_{A_{i_1}} \otimes \mathbf{1}_{A_{j_1}} \otimes \dots \otimes \mathbf{1}_{A_{j_n}}) \\ &= \delta_{m,n} \sum_{\pi \in \mathcal{P}_2(m \otimes m)} q^{\text{Cr}(\pi)} \int_{\mathbb{R}_+^n} d\mathbf{s} (\mathbf{1}_{A_{i_m}} \otimes \dots \otimes \mathbf{1}_{A_{i_1}})(\mathbf{s}^*) (\mathbf{1}_{A_{j_1}} \otimes \dots \otimes \mathbf{1}_{A_{j_m}})(\pi(\mathbf{s})_{m+1:2m}) \\ &= \delta_{m,n} \int_{\mathbb{R}_+^m} d\mathbf{s} (\mathbf{1}_{A_{i_1}} \otimes \dots \otimes \mathbf{1}_{A_{i_m}})(\mathbf{s}) P_q(\mathbf{1}_{A_{j_1}} \otimes \dots \otimes \mathbf{1}_{A_{j_m}})(\mathbf{s}) . \end{aligned}$$

Going back to (44), we get the desired conclusion for f and g , and the general result follows by linearity. \square

6. MULTIPLICATION FORMULAS IN q -WIENER CHAOS

With the formalism of Section 5 in hand, we can now state our main result about q -Wiener chaos, namely the extension, to every $q \in [0, 1)$, of the full Wick product formula:

Theorem 6.1. *Let $n_1, \dots, n_r \geq 1$ and for every $i \in \{1, \dots, r\}$, let $f_i \in L^2(\mathbb{R}_+^{n_i})$. Then, with the notations of Section 5.1, it holds that*

$$I_{n_1}^q(f_1) \dots I_{n_r}^q(f_r) = \sum_{\pi \in \mathcal{P}_{\leq 2}(n_1 \otimes \dots \otimes n_r)} q^{\text{Cr}(\pi)} I^q \left(\int_{\pi} f_1 \otimes \dots \otimes f_r \right) . \quad (45)$$

By letting q tend to 1 in (45) (at least at some heuristic level), we indeed recover the exact expression of the classical full Wick product formula for the standard Bm (see [12, Theorem 7.33]). Observe also that, as an immediate consequence of (45) and the fact that $\varphi(I_m^q(f)) = 0$ for $m \geq 1$, we recover the result of [6, Theorem 2.7]:

Corollary 6.2. *In the setting of Theorem 6.1, it holds that*

$$\varphi(I_{n_1}^q(f_1) \dots I_{n_r}^q(f_r)) = \sum_{\pi \in \mathcal{P}_2(n_1 \otimes \dots \otimes n_r)} q^{\text{Cr}(\pi)} \int_{\pi} f_1 \otimes \dots \otimes f_r .$$

The rest of this section is devoted to the proof of Theorem 6.1, which will consist in a natural three-step procedure: we first show the formula when $(r = 2, n_1 = 1, n_2 \geq 1)$, then extend the result to the case where $(r = 2, n_1, n_2 \geq 1)$, and finally turn to the general situation. At each step, our strategy will actually be based on the (non-trivial) q -extension of the classical arguments used in the commutative framework.

Proposition 6.3. *For all $f \in L^2(\mathbb{R}_+)$ and $g \in L^2(\mathbb{R}_+^n)$, it holds that*

$$I_1^q(f) I_n^q(g) = I_{n+1}^q(f \otimes g) + I_{n-1}^q(f \otimes_1^q g) . \quad (46)$$

Proof. For any interval A , let us write $X(A)$ for $X(\mathbf{1}_A)$. Using bilinearity and a density argument, it is readily checked that we can focus on the following situation:

$$f = \mathbf{1}_A \quad \text{and} \quad g = \mathbf{1}_{A_1} \otimes \cdots \otimes \mathbf{1}_{A_{k-1}} \otimes \mathbf{1}_B \otimes \mathbf{1}_{A_k} \otimes \cdots \otimes \mathbf{1}_{A_{n-1}},$$

where $k \in \{1, \dots, n\}$, the intervals A_i are disjoint and A, B are both intervals disjoint from the A_i 's such that $A = B$ or $A \cap B = \emptyset$.

If $A \cap B = \emptyset$, then clearly $f \otimes_1^q g = 0$ and the formula is trivially satisfied, so we assume from now on that $A = B$. Then, denoting by $\mu(A)$ the length of A , it is easy to see (as in Figure 1) that $f \otimes_1^q g = q^{k-1} \mu(A) \mathbf{1}_{A_1} \otimes \cdots \otimes \mathbf{1}_{A_{n-1}}$ and accordingly $I_{n-1}^q(f \otimes_1^q g) = q^{k-1} \mu(A) X(A_1) \cdots X(A_{n-1})$.

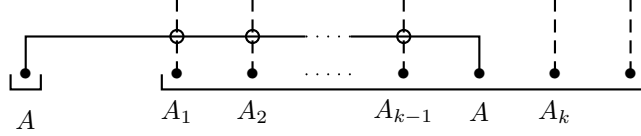


FIGURE 1. The only partition with non-zero contribution in the computation of $f \otimes_1^q g$.

For $\varepsilon > 0$, pick disjoint intervals B_1, \dots, B_l such that $A = B_1 \cup \dots \cup B_l$ and $\mu(B_i) < \varepsilon$. Then write

$$\begin{aligned} I_1(f)I_n(g) &= X(A)X(A_1) \cdots X(A_{k-1})X(A)X(A_k) \cdots X(A_{n-1}) \\ &= \sum_{i \neq j} X(B_i)X(A_1) \cdots X(A_{k-1})X(B_j)X(A_k) \cdots X(A_{n-1}) \\ &\quad + \sum_{i=1}^l \{X(B_i)X(A_1) \cdots X(A_{k-1})X(B_i)X(A_k) \cdots X(A_{n-1}) - q^{k-1} \mu(B_i)X(A_1) \cdots X(A_{n-1})\} \\ &\quad + q^{k-1} \mu(A)X(A_1) \cdots X(A_{n-1}) \\ &= I_{n+1}^q(h_\varepsilon) + \sum_{i=1}^l R_{\varepsilon,i} + I_{n-1}^q(f \otimes_1^q g), \end{aligned}$$

where we have set

$$h_\varepsilon := \sum_{i \neq j} \mathbf{1}_{B_i} \otimes \mathbf{1}_{A_1} \otimes \cdots \otimes \mathbf{1}_{A_{k-1}} \otimes \mathbf{1}_{B_j} \otimes \mathbf{1}_{A_k} \otimes \cdots \otimes \mathbf{1}_{A_{n-1}}.$$

At this point, observe that due to (43), one has

$$\|I_{n+1}^q(h_\varepsilon) - I_{n+1}^q(f \otimes g)\|_{L^2(\varphi)} \leq (n+1)! \|h_\varepsilon - f \otimes g\|_{L^2(\mathbb{R}_+^n)},$$

which tends to 0 as $\varepsilon \rightarrow 0$. Therefore, it remains us to check that $\|\sum_{i=1}^l R_{\varepsilon,i}\|_{L^2(\varphi)} \rightarrow 0$ as $\varepsilon \rightarrow 0$, and to this end, decompose $\|\sum_{i=1}^l R_{\varepsilon,i}\|_{L^2(\varphi)}^2$ as

$$\left\| \sum_{i=1}^l R_{\varepsilon,i} \right\|_{L^2(\varphi)}^2 = \sum_{i \neq j} \varphi(R_{\varepsilon,i}^* R_{\varepsilon,j}) + \sum_{i=1}^l \varphi(R_{\varepsilon,i}^* R_{\varepsilon,i}). \quad (47)$$

When $i \neq j$, it turns out that $\varphi(R_{\varepsilon,i}^* R_{\varepsilon,j}) = 0$, as a consequence of the following readily-checked relations (see Figure 2 for an illustration of the first one):

$$\begin{aligned} &\varphi([X(A_{n-1}) \cdots X(A_k)X(B_i)X(A_{k-1}) \cdots X(A_1)X(B_i)] \cdot \\ &\quad [X(B_j)X(A_1) \cdots X(A_{k-1})X(B_j)X(A_k) \cdots X(A_{n-1})]) \\ &= q^{2(k-1)} \mu(B_i) \mu(B_j) \mu(A_1) \cdots \mu(A_{n-1}), \end{aligned} \quad (48)$$

$$\begin{aligned} & \varphi([X(A_{n-1}) \cdots X(A_k)X(B_i)X(A_{k-1}) \cdots X(A_1)X(B_i)] \cdot [X(A_1) \cdots X(A_{n-1})]) \\ &= q^{k-1} \mu(B_i) \mu(A_1) \cdots \mu(A_{n-1}) , \end{aligned}$$

and

$$\varphi([X(A_{n-1}) \cdots X(A_1)] \cdot [X(A_1) \cdots X(A_{n-1})]) = \mu(A_1) \cdots \mu(A_{n-1}) .$$

As for the second summand in (47), it is easy to see that $\varphi(R_{\varepsilon,i}^* R_{\varepsilon,i}) \leq c \mu(B_i)^2$, so

$$0 \leq \sum_{i=1}^l \varphi(R_{\varepsilon,i}^* R_{\varepsilon,i}) \leq c \varepsilon \mu(A) ,$$

which, by letting ε tend to zero, completes the proof of our statement. □

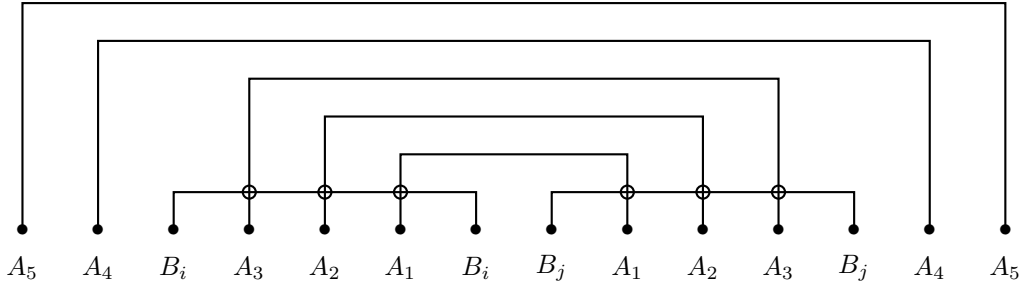


FIGURE 2. The only partition with non-zero contribution in (48) (here, $k = 4, n = 6$).

Let us now extend (46) to any function $f \in L^2(\mathbb{R}_+^m)$, $m \geq 1$:

Proposition 6.4. *For all $f \in L^2(\mathbb{R}_+^m)$ and $g \in L^2(\mathbb{R}_+^n)$, it holds that*

$$I_m^q(f) I_n^q(g) = \sum_{k=0}^{m \wedge n} I_{m+n-2k}^q(f \otimes_k^q g) . \quad (49)$$

The key ingredient towards (49) is the following recursion formula satisfied by q -contractions:

Lemma 6.5. *Fix $m \leq n - 1$ and let $g \in L^2(\mathbb{R}_+^n)$, $f_1 := \mathbf{1}_{A_1}$, $f_2 := \mathbf{1}_{A_2} \otimes \cdots \otimes \mathbf{1}_{A_{m+1}}$, for disjoint intervals (A_i) . Then the following relations hold true:*

$$(f_1 \otimes f_2) \otimes_{m+1}^q g = f_1 \otimes_1^q (f_2 \otimes_m^q g),$$

and for every $k = 1, \dots, m$,

$$(f_1 \otimes f_2) \otimes_k^q g = f_1 \otimes (f_2 \otimes_k^q g) + f_1 \otimes_1^q (f_2 \otimes_{k-1}^q g) .$$

Proof. Given $\mathbf{t} := (t_1, \dots, t_n) \in \mathbb{R}_+^n$ and $\ell \leq m \in \{1, \dots, n\}$, let us set $\mathbf{t}_{\ell::m} := (t_\ell, t_{\ell+1}, \dots, t_m)$. Using this notation, one has, for all $k \in \{0, \dots, m\}$ and $\mathbf{t} \in \mathbb{R}_+^{m+n-2k-1}$,

$$\begin{aligned} & [f_1 \otimes_1^q (f_2 \otimes_k^q g)](\mathbf{t}) \\ &= \sum_{\ell=1}^{m+n-2k} q^{\ell-1} \int_{\mathbb{R}_+} ds f_1(s) (f_2 \otimes_k^q g)(\mathbf{t}_{1::\ell-1}, s, \mathbf{t}_{\ell::m+n-2k-1}) \\ &= \sum_{\ell=1}^{m+n-2k} \sum_{\pi \in \mathcal{P}_{\leq 2}^k(m \otimes n)} q^{\ell-1 + \text{Cr}(\pi)} \int_{\mathbb{R}_+} ds f_1(s) (f_2 \otimes_\pi g)(\mathbf{t}_{1::\ell-1}, s, \mathbf{t}_{\ell::m+n-2k-1}) . \end{aligned}$$

Since f_1 and f_2 have disjoint supports, it is easy to see that if $\ell \in \{1, \dots, m-k\}$ then for every $\pi \in \mathcal{P}_{\leq 2}^k(m \otimes n)$,

$$\int_{\mathbb{R}^+} ds f_1(s)(f_2 \otimes_\pi g)(\mathbf{t}_{1::\ell-1}, s, \mathbf{t}_{\ell::m+n-2k-1}) = 0,$$

and accordingly the above formula reduces to

$$\begin{aligned} & [f_1 \otimes_1^q (f_2 \otimes_k^q g)](\mathbf{t}) \\ &= \sum_{\ell=m-k+1}^{m+n-2k} \sum_{\pi \in \mathcal{P}_{\leq 2}^k(m \otimes n)} q^{\ell-1+\text{Cr}(\pi)} \int_{\mathbb{R}^+} ds f_1(s)(f_2 \otimes_\pi g)(\mathbf{t}_{1::\ell-1}, s, \mathbf{t}_{\ell::m+n-2k-1}). \end{aligned} \quad (50)$$

Now observe that there is a one-to-one correspondance between pairs $(\ell, \pi) \in \{m-k+1, \dots, m+n-2k\} \times \mathcal{P}_{\leq 2}^k(m \otimes n)$ and partitions $\pi' \in \mathcal{P}_{\leq 2}^{k+1}((m+1) \otimes n)$ not containing the singleton (1). Namely, given such a pair (ℓ, π) , we can construct π' along the following two-step procedure (see Figure 3):

- let the k interactions between the blocks $\{2, \dots, m+1\}$ and $\{m+2, \dots, m+n\}$ be governed by π ;
- connect the point 1 with the ℓ -th unpaired point of $\{2, \dots, m+n+1\}$, when counting from the left to the right (in particular, the right-end point of this pair necessarily belongs to $\{m+2, \dots, m+n+1\}$ due to $\ell \geq m-k+1$).

With these notations, it is readily checked (see again Figure 3) that $\text{Cr}(\pi') = \text{Cr}(\pi) + (\ell - 1)$ and

$$\int_{\mathbb{R}^+} ds f_1(s)(f_2 \otimes_\pi g)(\mathbf{t}_{1::\ell-1}, s, \mathbf{t}_{\ell::m+n-2k-1}) = [(f_1 \otimes f_2) \otimes_{\pi'} g](\mathbf{t}).$$

Going back to (50), we deduce that

$$f_1 \otimes_1^q (f_2 \otimes_k^q g) = \sum_{\substack{\pi \in \mathcal{P}_{\leq 2}^{k+1}((m+1) \otimes n) \\ (1) \notin \pi}} q^{\text{Cr}(\pi)} (f_1 \otimes f_2) \otimes_\pi g. \quad (51)$$

When $k = m$, (51) reduces to

$$f_1 \otimes_1^q (f_2 \otimes_m^q g) = \sum_{\pi \in \mathcal{P}_{\leq 2}^{k+1}((m+1) \otimes n)} q^{\text{Cr}(\pi)} (f_1 \otimes f_2) \otimes_\pi g = (f_1 \otimes f_2) \otimes_{m+1}^q g,$$

which corresponds to the first claim of our statement. Then, for $k \in \{0, \dots, m-1\}$, one has

$$\begin{aligned} (f_1 \otimes f_2) \otimes_{k+1}^q g &= \sum_{\pi \in \mathcal{P}_{\leq 2}^{k+1}((m+1) \otimes n)} q^{\text{Cr}(\pi)} (f_1 \otimes f_2) \otimes_\pi g \\ &= \sum_{\substack{\pi \in \mathcal{P}_{\leq 2}^{k+1}((m+1) \otimes n) \\ (1) \in \pi}} q^{\text{Cr}(\pi)} (f_1 \otimes f_2) \otimes_\pi g + \sum_{\substack{\pi \in \mathcal{P}_{\leq 2}^{k+1}((m+1) \otimes n) \\ (1) \notin \pi}} q^{\text{Cr}(\pi)} (f_1 \otimes f_2) \otimes_\pi g \\ &= f_1 \otimes (f_2 \otimes_{k+1}^q g) + \sum_{\substack{\pi \in \mathcal{P}_{\leq 2}^{k+1}((m+1) \otimes n) \\ (1) \notin \pi}} q^{\text{Cr}(\pi)} (f_1 \otimes f_2) \otimes_\pi g, \end{aligned}$$

and we can conclude by using (51) again. \square

Proof of Proposition 6.4. Assume that formula (49) holds true for all $m \leq n$, $f \in L^2(\mathbb{R}_+^m)$ and $g \in L^2(\mathbb{R}_+^n)$. Then for $m \geq n$, it holds that

$$I_m^q(f) I_n^q(g) = (I_n^q(g^*) I_m^q(f^*))^* = \left(\sum_{k=0}^{m \wedge n} I_{n+m-2k}^q(g^* \otimes_k^q f^*) \right)^* = \sum_{k=0}^{m \wedge n} I_{m+n-2k}^q(f \otimes_k^q g),$$

where we have used Lemma 5.5 to derive the last equality.

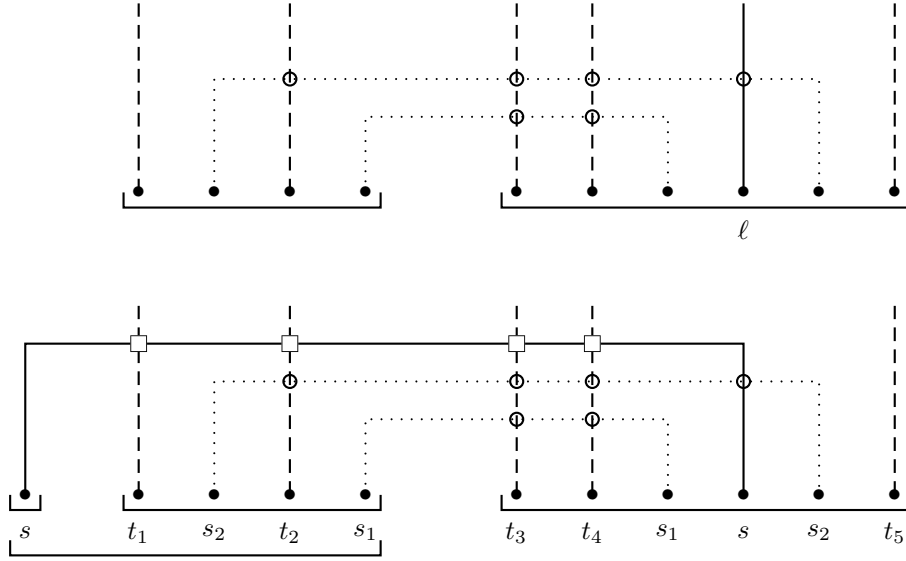


FIGURE 3. Construction of π' (second line) from π and the ℓ -th unpaired position (first line). Here, $m = 4, n = 6, r = 2$ and $\ell = 5$.

Therefore, we can stick to an induction procedure on $m \geq 1$ for $n \geq m$. If $m = 1$, then (49) is nothing but the result of Proposition 6.3. Assume that the decomposition holds true for some $m \geq 1$ and every $n \geq m$, and let $n \geq m + 1$. By a density argument, we can take $f \in L^2(\mathbb{R}_+^{m+1})$ of the form $f = f_1 \otimes f_2$ with $f_1 = \mathbf{1}_{A_1}$ and $f_2 = \mathbf{1}_{A_2} \otimes \cdots \otimes \mathbf{1}_{A_{m+1}}$, for disjoint intervals (A_i) . Then $I_{m+1}^q(f) = I_1^q(f_1)I_m^q(f_2)$ and

$$\begin{aligned}
 & I_{m+1}^q(f)I_n^q(g) \\
 &= I_1^q(f_1) \cdot [I_m^q(f_2)I_n^q(g)] \\
 &= \sum_{k=0}^m I_1^q(f_1)I_{m+n-2k}^q(f_2 \otimes_k^q g) \quad (\text{by the induction hypothesis}) \\
 &= \sum_{k=0}^m [I_{m+n+1-2k}^q(f_1 \otimes (f_2 \otimes_k^q g)) + I_{m+n-1-2k}^q(f_1 \otimes_1^q (f_2 \otimes_k^q g))] \quad (\text{by Proposition 6.3}) \\
 &= I_{m+n+1}^q(f_1 \otimes f_2 \otimes g) + \sum_{k=1}^m I_{m+n+1-2k}^q(f_1 \otimes (f_2 \otimes_k^q g) + f_1 \otimes_1^q (f_2 \otimes_{k-1}^q g)) \\
 &\quad + I_{n-m-1}^q(f_1 \otimes_1^q (f_2 \otimes_m^q g)) .
 \end{aligned}$$

The conclusion immediately follows from the two identities exhibited in Lemma 6.5. □

We can finally turn to the proof of the general formula.

Proof of Theorem 6.1. We proceed by induction on $r \geq 1$. For $r = 1$, the result only amounts to saying that $q^{\text{Cr}(\pi)} \int_{\pi} f_1 = f_1$ when $\pi = \{(1), \dots, (n_1)\}$, which is obvious. Assume that the relation holds true

up to $r - 1$ and let $f_i \in L^2(\mathbb{R}_+^{n_i})$ for $i = 1, \dots, r$. Then, using Proposition 6.4, we have

$$\begin{aligned}
& I_{n_1}^q(f_1) \cdots I_{n_r}^q(f_r) \\
&= \sum_{k=0}^{n_1 \wedge n_2} \sum_{\pi_1 \in \mathcal{P}_{\leq 2}^k(n_1 \otimes n_2)} q^{\text{Cr}(\pi_1)} I^q(f_1 \otimes_{\pi_1} f_2) I_{n_3}^q(f_3) \cdots I_{n_r}^q(f_r) \\
&= \sum_{k=0}^{n_1 \wedge n_2} \sum_{\pi_1 \in \mathcal{P}_{\leq 2}^k(n_1 \otimes n_2)} \sum_{\pi_2 \in \mathcal{P}_{\leq 2}((n_1 + n_2 - 2k) \otimes n_3 \otimes \cdots \otimes n_r)} q^{\text{Cr}(\pi_1) + \text{Cr}(\pi_2)} \\
&\quad I^q \left(\int_{\pi_2} [f_1 \otimes_{\pi_1} f_2] \otimes f_3 \otimes \cdots \otimes f_r \right). \tag{52}
\end{aligned}$$

At this point, and in a similar way as in the proof of Lemma 6.5, observe that there is a one-to-one correspondance between the set of triplets (k, π_1, π_2) in the above sum and the set of partitions $\pi \in \mathcal{P}_{\leq 2}(n_1 \otimes \cdots \otimes n_r)$. Namely, given such a triplet (k, π_1, π_2) , we can construct π along the following 2-step procedure (see Figure 4):

- the first two blocks $\{1, \dots, n_1\}$ and $\{n_1 + 1, \dots, n_1 + n_2\}$ are connected by k pairs, and these interactions are described through π_1 ;
- then π_2 governs the interactions between the $n_1 + n_2 - 2k$ unpaired points of $\{1, \dots, n_1 + n_2\}$ and the set $\{n_1 + n_2 + 1, \dots, n_1 + \cdots + n_r\}$.

With these notations, it only remains to observe (see Figure 4 again) that $\text{Cr}(\pi) = \text{Cr}(\pi_1) + \text{Cr}(\pi_2)$ and

$$\int_{\pi_2} [f_1 \otimes_{\pi_1} f_2] \otimes f_3 \otimes \cdots \otimes f_r = \int_{\pi} f_1 \otimes f_2 \otimes f_3 \otimes \cdots \otimes f_r.$$

Going back to (52), we get the desired conclusion. \square

7. MULTIPLICATION IN THE FULLY-SYMMETRIC CASE

As a conclusion to our study, let us elaborate on the specific situation where the kernels involved within the multiple integrals under consideration are given by *fully-symmetric* functions. Recall that a function $f : \mathbb{R}_+^n \rightarrow \mathbb{R}$ is said to be fully-symmetric if for all times $t_1, \dots, t_n \in \mathbb{R}_+$ and every permutation σ of $\{1, \dots, n\}$, one has $f(t_1, \dots, t_n) = f(t_{\sigma(1)}, \dots, t_{\sigma(n)})$.

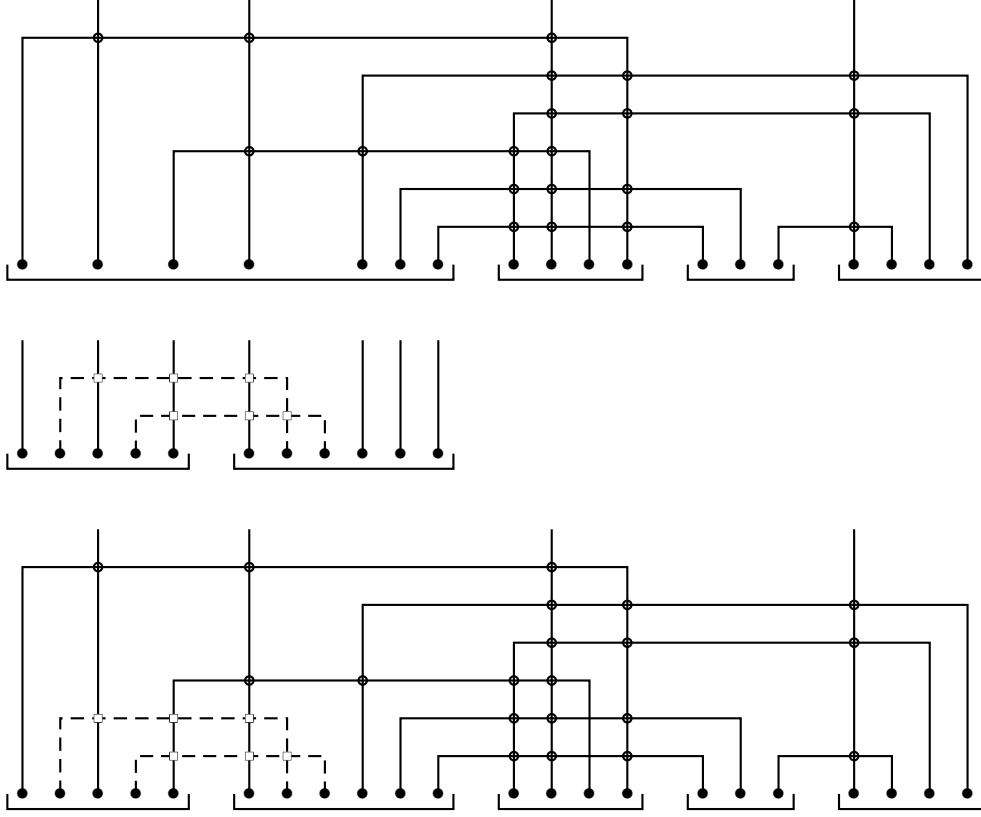
It is important to notice here that, contrary to the classical commutative case (with multiple integrals generated by a standard Bm), the fully-symmetric assumption must be regarded as highly restrictive in the non-commutative situation. Otherwise stated, considering any generic function $f \in L^2(\mathbb{R}_+^n)$ ($n \geq 2$) and its symmetrization

$$\tilde{f}(t_1, \dots, t_n) := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} f(t_{\sigma(1)}, \dots, t_{\sigma(n)}),$$

there is no reason in general for the two multiple integrals $I_n^q(f)$ and $I_n^q(\tilde{f})$ to be equal, which is easy to see from the very definition (42) of the multiple integrals and the possible non-commutativity of the components of X .

7.1. Multiplication formula. Given two fully-symmetric functions $f_1 \in L^2(\mathbb{R}_+^{n_1})$, $f_2 \in L^2(\mathbb{R}_+^{n_2})$ and a parameter $k \in \{1, \dots, n_1 \wedge n_2\}$, it is readily checked that the contraction $f_1 \otimes_{\pi} f_2$ (see Definition 5.3) does not depend on the choice of $\pi \in \mathcal{P}_{\leq 2}^k(n_1 \otimes n_2)$: we denote by $f_1 \otimes_k f_2$ this common value. For $q \in [0, 1)$, the q -contraction of f_1 and f_2 (see again Definition 5.3) is then equal to $f_1 \otimes_k^q f_2 = C_{n_1, n_2}^{(k)}(q) f \otimes_k g$, where

$$C_{n_1, n_2}^{(k)}(q) := \sum_{\pi \in \mathcal{P}_{\leq 2}^k(n_1 \otimes n_2)} q^{\text{Cr}(\pi)}.$$

FIGURE 4. Construction of π (third line) from π_1 (second line) and π_2 (first line).

We propose to show that the latter coefficients can be conveniently expressed in terms of standard q -binomial coefficients. To this end, and for the sake of completeness, let us first recall the definition of the classical combinatorial coefficients associated with q -combinatorics:

$$[n]_q = 1 + q + \cdots + q^{n-1} \quad , \quad [n]_q! := [n]_q [n-1]_q \cdots [1]_q \quad ,$$

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!} \quad , \quad \binom{n}{n_1, \dots, n_p}_q := \frac{[n]_q!}{[n_1]_q! \cdots [n_p]_q!} \quad ,$$

for all $n_1, \dots, n_p \geq 0$ such that $n_1 + \cdots + n_p = n$.

Proposition 7.1. *For all $m, n \geq 0$ and $k \in \{0, \dots, m \wedge n\}$, it holds that*

$$C_{m,n}^{(k)}(q) = [k]_q! \binom{n}{k}_q \binom{m}{k}_q . \quad (53)$$

Injecting (53) into (49) immediately gives rise to the following combinatorial description of the product of any two multiple integrals build upon fully-symmetric kernels. Note that, when compared for instance with [15, Proposition 1.1.3], this formulation makes the transition with the classical commutative case even more clear.

Theorem 7.2. *Let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be fully-symmetric functions. Then it holds that*

$$I_n^q(f)I_m^q(g) = \sum_{k=0}^{n \wedge m} [k]_q! \binom{n}{k}_q \binom{m}{k}_q I_{n+m-2k}^q(f \otimes_k g).$$

Before we turn to the proof of (53), observe that this relation extends the well-known formula

$$\sum_{\pi \in \mathcal{P}_2(n \otimes n)} q^{\text{Cr}(\pi)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} = [n]_q!, \quad (54)$$

where \mathfrak{S}_n refers to the group of permutations of $\{1, \dots, n\}$ and $\text{inv}(\sigma)$ to the number of inversions in σ . In order to extend (54) into (53), we will essentially rely on the following combinatorial result:

Lemma 7.3. *For all $n \geq k \geq 1$, it holds that*

$$\sum_{1 \leq i_1 < \dots < i_k \leq n} q^{i_1 + \dots + i_k} = q^{\frac{k(k+1)}{2}} \binom{n}{k}_q. \quad (55)$$

Proof. By induction on $k \geq 1$. For $k = 1$, one has indeed, for all $n \geq 1$, $\sum_{1 \leq i \leq n} q^i = q[n]_q$. Assume that the relation holds true up to some fixed $k \geq 1$, for all $n \geq k$. Then for $n \geq k$, one has

$$\begin{aligned} \sum_{1 \leq i_1 < \dots < i_{k+1} \leq n+1} q^{i_1 + \dots + i_{k+1}} &= \sum_{1 \leq i_1 \leq n+1-k} q^{i_1} \sum_{i_1+1 \leq i_2 < \dots < i_{k+1} \leq n+1} q^{i_2 + \dots + i_{k+1}} \\ &= \sum_{1 \leq i \leq n+1-k} q^{(k+1)i} \sum_{1 \leq i_1 < \dots < i_k \leq n+1-i} q^{i_1 + \dots + i_k} \\ &= q^{\frac{k(k+1)}{2}} \sum_{1 \leq i \leq n+1-k} q^{(k+1)i} \binom{n+1-i}{k}_q \\ &= q^{\frac{k(k+1)}{2}} \sum_{k \leq i \leq n} q^{(k+1)(n+1-i)} \binom{i}{k}_q. \end{aligned}$$

Let us now prove by induction on $n \geq k$ that

$$\sum_{k \leq i \leq n} q^{(k+1)(n+1-i)} \binom{i}{k}_q = q^{k+1} \binom{n+1}{k+1}_q, \quad \text{i.e.,} \quad \sum_{k \leq i \leq n} q^{(k+1)(n-i)} \binom{i}{k}_q = \binom{n+1}{k+1}_q.$$

For $n = k$, the relation is obviously satisfied. Then, assuming that it holds true for some $n \geq k$, we get

$$\begin{aligned} \sum_{k \leq i \leq n+1} q^{(k+1)(n+1-i)} \binom{i}{k}_q &= q^{k+1} \sum_{k \leq i \leq n} q^{(k+1)(n-i)} \binom{i}{k}_q + \binom{n+1}{k}_q \\ &= q^{k+1} \binom{n+1}{k+1}_q + \binom{n+1}{k}_q = \binom{n+2}{k+1}_q. \end{aligned}$$

This achieves the proof of (55). \square

Proof of Proposition 7.1. Observe that any partition $\pi \in \mathcal{P}_{\leq 2}^k(m \otimes n)$ can be entirely described through two elements (see Figure 5):

- the positions $\{m+1-i_k < \dots < m+1-i_1\}$ in $\{1, \dots, m\}$ (resp. $\{m+j_1 < \dots < m+j_k\}$ in $\{m+1, \dots, m+n\}$) corresponding to the left-end (resp. right-end) points of the pairs in π ;
- the interactions in π between these paired points, which can be summed through a unique $\pi' \in \mathcal{P}_2(k \otimes k)$.

Setting $\mathbf{i} := (i_1, \dots, i_k)$ and $\mathbf{j} := (j_1, \dots, j_k)$, the difference $\text{Cr}(\mathbf{i}, \mathbf{j}) := \text{Cr}(\pi) - \text{Cr}(\pi')$ is then given by the number of crossings between a pair and a singleton in π . We can easily compute this quantity as follows (see again Figure 5):

$$\begin{aligned} \text{Cr}(\mathbf{i}, \mathbf{j}) &= [k \times (i_1 - 1) + (k-1) \times (i_2 - i_1 - 1) + \dots + 1 \times (i_k - i_{k-1} - 1) + 0 \times (m - i_k)] \\ &\quad + [k \times (j_1 - 1) + (k-1) \times (j_2 - j_1 - 1) + \dots + 1 \times (j_k - j_{k-1} - 1) + 0 \times (n - j_k)] , \end{aligned}$$

which in fact reduces to

$$\text{Cr}(\mathbf{i}, \mathbf{j}) = \sum_{l=1}^k (i_l + j_l) - k(k+1) .$$

As a consequence,

$$\begin{aligned} \sum_{\pi \in \mathcal{P}_{\leq 2}^k(m \otimes n)} q^{\text{Cr}(\pi)} &= \left(\sum_{\pi' \in \mathcal{P}_2(k \otimes k)} q^{\text{Cr}(\pi')} \right) \cdot \left(\sum_{\substack{1 \leq i_1 < \dots < i_k \leq m \\ 1 \leq j_1 < \dots < j_k \leq n}} q^{\text{Cr}(\mathbf{i}, \mathbf{j})} \right) \\ &= q^{-k(k+1)} \left(\sum_{\pi' \in \mathcal{P}_2(k \otimes k)} q^{\text{Cr}(\pi')} \right) \cdot \left(\sum_{1 \leq i_1 < \dots < i_k \leq m} q^{i_1 + \dots + i_k} \right) \cdot \left(\sum_{1 \leq j_1 < \dots < j_k \leq n} q^{j_1 + \dots + j_k} \right) , \end{aligned}$$

and we can conclude by applying formula (54) and Lemma 7.3. \square

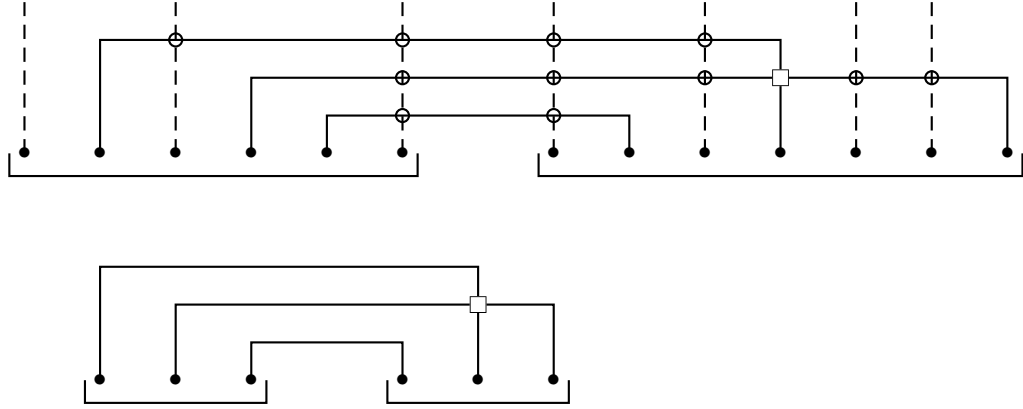


FIGURE 5. A partition $\pi \in \mathcal{P}_{\leq 2}^3(6 \times 7)$ (first line) with its associated partition $\pi' \in \mathcal{P}_2(3 \otimes 3)$ (second line). With the notations of the proof, the positions of the pairs in π correspond to $i_1 = 2, i_2 = 3, i_3 = 5$, and $j_1 = 2, j_2 = 4, j_3 = 7$.

7.2. Stochastic dynamics of the q -Hermite martingales. A well-known example of multiple integrals involving fully-symmetric kernels is provided by the sequence of the q -Hermite martingales

$$M_n^{(q)}(t) := I_n^q(\mathbf{1}_{[0,t]}^{\otimes n}) , \quad n \geq 1 ,$$

which corresponds to the q -analog of the classical sequence of Hermite martingales derived from Hermite polynomials. The study of the martingale properties of $M_n^{(q)}$, as well as its connections with the q -Hermite polynomials, can be found in [3, Proposition 2.9 and Corollary 4.7].

What we propose to do here is to use the results of the previous sections (and especially identity (53)) so as to complete the program initiated in [8, Section 4.2] regarding the stochastic dynamics of the sequence $(M_n^{(q)})$. Let us first recall that in the classical commutative framework, the sequence $(M_n^{(1)})$ of Hermite martingales is known to be governed by the formula (denoting by $X^{(1)}$ the standard Bm)

$$M_{n+1}^{(1)}(t) = (n+1) \int_0^t M_n^{(1)}(s) dX_s^{(1)} , \quad (56)$$

while in the free situation, that is when $q = 0$, the following relation can be found in [1]

$$M_{n+1}^{(0)}(t) = \sum_{0 \leq k \leq n} \int_0^t M_k^{(0)}(s) dX_s^{(0)} M_{n-k}^{(0)}(s) . \quad (57)$$

In order to express our interpolation result between these two formulas, let us consider the family of kernels defined as

$$h_{m,t}^\ell(s_1, \dots, s_\ell) := \mathbf{1}_{[0,t]}^{\otimes \ell}(s_1, \dots, s_\ell) \prod_{i \neq m} \mathbf{1}_{\{s_i < s_m\}} \quad , \quad \ell, m \geq 1 \quad , \quad t, s_1, \dots, s_\ell \geq 0 \quad .$$

Proposition 7.4. *For all $m, n \geq 0$ and $t \geq 0$, it holds that*

$$I_{m+n+1}^q(h_{m+1,t}^{m+n+1}) = \sum_{\ell=0}^{m \wedge n} (-1)^\ell q^{\frac{\ell(\ell+1)}{2}} C_{m,n}^{(\ell)}(q) \int_0^t s^\ell M_{m-\ell}^{(q)}(s) dX_s^{(q)} M_{n-\ell}^{(q)}(s) \quad , \quad (58)$$

where the integral in the right-hand side is understood in Itô's sense (see Definition 3).

To see that identity (58) indeed provides us with the desired interpolation between (56) and (57), it suffices to observe that, on the one hand,

$$M_{n+1}^{(1)}(t) = I_n^1(\mathbf{1}_{[0,t]}^{\otimes n}) = (n+1)I_{n+1}^1(h_{1,t}^{n+1}) \quad ,$$

which gives (56) by applying (58) with $m = 0$. On the other hand, by writing

$$M_{n+1}^{(q)}(t) = I_n^q(\mathbf{1}_{[0,t]}^{\otimes n}) = \sum_{k=0}^n I_{(n-k)+k+1}^q(h_{k+1,t}^{(n-k)+k+1})$$

and then applying (58) to each summand, we immediately recover (57) for $q = 0$, and more generally:

Corollary 7.5. *For all $n \geq 0$ and $t \geq 0$, it holds that*

$$M_{n+1}^{(q)}(t) = \sum_{k=0}^n \sum_{\ell=0}^{k \wedge (n-k)} (-1)^\ell q^{\frac{\ell(\ell+1)}{2}} C_{k,n-k}^{(\ell)}(q) \int_0^t s^\ell M_{k-\ell}^{(q)}(s) dX_s^{(q)} M_{n-k-\ell}^{(q)}(s) \quad ,$$

where the integral in the right-hand side is understood in Itô's sense.

Proof of Proposition 7.4. For the sake of clarity, we fix q and t for the whole proof and drop the dependence on these two parameters in the notation, that is we write $C_{m,n}^{(k)}$ for $C_{m,n}^{(k)}(q)$, h_m^ℓ for $h_{m,t}^\ell$, and so on. Besides, let us set, for all $\ell, p \geq 0$, $m, n \geq 1$ and $s_1, \dots, s_\ell \geq 0$,

$$h_m^{\ell,p}(s_1, \dots, s_\ell) := s_m^p \mathbf{1}_{[0,t]}^{\otimes \ell}(s_1, \dots, s_\ell) \prod_{i \neq m} \mathbf{1}_{\{s_i < s_m\}} \quad , \quad J_{m,n}^\ell := \int_0^t s^\ell M_m(s) dX_s M_n(s) \quad ,$$

noting in particular that $h_m^\ell = h_m^{\ell,0}$. With these notations in hand, a straightforward application of [8, Proposition 4.2] yields that

$$J_{m,n}^p = I_{m+n+1}(h_{m+1}^{m+n+1,p}) + \sum_{\ell_1=1}^{m \wedge n} q^{\ell_1} C_{m,n}^{(\ell_1)} I_{m+n+1-2\ell_1}(h_{m+1-\ell_1}^{m+n+1-2\ell_1,p+\ell_1}) \quad .$$

Therefore,

$$\begin{aligned} & I_{m+n+1}(h_{m+1}^{m+n+1}) \\ &= J_{m,n}^0 - \sum_{\ell_1=1}^{m \wedge n} q^{\ell_1} C_{m,n}^{(\ell_1)} I_{m+n+1-2\ell_1}(h_{m+1-\ell_1}^{m+n+1-2\ell_1,\ell_1}) \\ &= J_{m,n}^0 - \sum_{\ell_1=1}^{m \wedge n} q^{\ell_1} C_{m,n}^{(\ell_1)} \left[J_{m-\ell_1,n-\ell_1}^{\ell_1} - \sum_{\ell_2=1}^{(m \wedge n)-\ell_1} q^{\ell_2} C_{m-\ell_1,n-\ell_1}^{(\ell_2)} \right. \\ & \quad \left. I_{m+n+1-2(\ell_1+\ell_2)}(h_{m+1-(\ell_1+\ell_2)}^{m+n+1-2(\ell_1+\ell_2),\ell_1+\ell_2}) \right] \end{aligned}$$

and by repeating the procedure, we end up with

$$\begin{aligned} I_{m+n+1}(h_{m+1}^{m+n+1}) &= J_{m,n}^0 + \sum_{(\ell_1, \dots, \ell_p) \in \mathcal{G}_{m,n}} (-1)^p q^{\ell_1 + \dots + \ell_p} [C_{m,n}^{(\ell_1)} C_{m-\ell_1, n-\ell_1}^{(\ell_2)} \cdots C_{m-(\ell_1 + \dots + \ell_{p-1}), n-(\ell_1 + \dots + \ell_{p-1})}^{(\ell_p)}] \\ &\quad J_{m-(\ell_1 + \dots + \ell_p), n-(\ell_1 + \dots + \ell_p)}^{\ell_1 + \dots + \ell_p}, \end{aligned}$$

where

$$\mathcal{G}_{m,n} := \{(\ell_1, \dots, \ell_p) : p \geq 1, 1 \leq \ell_1 \leq m \wedge n, 1 \leq \ell_{i+1} \leq (m \wedge n) - (\ell_1 + \dots + \ell_i)\}.$$

Observe that this expression can also be written as

$$\begin{aligned} I_{m+n+1}(h_{m+1}^{m+n+1}) &= J_{m,n} + \sum_{\ell=1}^{m \wedge n} q^\ell J_{m-\ell, n-\ell}^\ell \sum_{p=1}^{\ell} (-1)^p \sum_{\substack{\ell_1 + \dots + \ell_p = \ell \\ \ell_i \geq 1}} [C_{m,n}^{(\ell_1)} C_{m-\ell_1, n-\ell_1}^{(\ell_2)} \cdots C_{m-(\ell_1 + \dots + \ell_{p-1}), n-(\ell_1 + \dots + \ell_{p-1})}^{(\ell_p)}]. \end{aligned}$$

Now, using identity (53), it is readily checked that for $\ell_1, \dots, \ell_p \in \{1, \dots, m \wedge n\}$ with $\ell_1 + \dots + \ell_p = \ell$, one has

$$C_{m,n}^{(\ell_1)} C_{m-\ell_1, n-\ell_1}^{(\ell_2)} \cdots C_{m-(\ell_1 + \dots + \ell_{p-1}), n-(\ell_1 + \dots + \ell_{p-1})}^{(\ell_p)} = C_{m,n}^{(\ell)} \binom{\ell}{\ell_1, \dots, \ell_p}_q,$$

and thus, using also the subsequent combinatorial Lemma 7.6, we obtain

$$I_{m+n+1}(h_{m+1}^{m+n+1}) = J_{m,n}^0 + \sum_{\ell=1}^{m \wedge n} (-1)^\ell q^{\frac{\ell(\ell+1)}{2}} C_{m,n}^{(\ell)} J_{m-\ell, n-\ell}^\ell,$$

which corresponds to the expected formula. □

Lemma 7.6. *For every $\ell \geq 1$, it holds that*

$$\sum_{p=1}^{\ell} (-1)^p \sum_{\substack{\ell_1 + \dots + \ell_p = \ell \\ \ell_i \geq 1}} \binom{\ell}{\ell_1, \dots, \ell_p}_q = (-1)^\ell q^{\frac{\ell(\ell-1)}{2}}.$$

Proof. By induction on $\ell \geq 1$. For $\ell = 1$, the relation is immediate. Assume that it holds true up to $\ell \geq 1$, and recall the classical recursion formula for the q -multinomial coefficients: if $\ell_1 + \dots + \ell_p = \ell + 1$, then

$$\binom{\ell+1}{\ell_1, \dots, \ell_p}_q = \sum_{m=1}^p q^{\ell_1 + \dots + \ell_{m-1}} \binom{\ell}{\ell_1, \dots, \ell_m - 1, \dots, \ell_p}_q.$$

Therefore,

$$\begin{aligned}
& \sum_{p=1}^{\ell+1} (-1)^p \sum_{\substack{\ell_1+\dots+\ell_p=\ell+1 \\ \ell_i \geq 1}} \binom{\ell+1}{\ell_1, \dots, \ell_p}_q \\
&= \sum_{p=1}^{\ell+1} (-1)^p \sum_{m=1}^p \sum_{\substack{\ell_1+\dots+(\ell_m-1)+\dots+\ell_p=\ell \\ \ell_i \geq 1}} q^{\ell_1+\dots+\ell_{m-1}} \binom{\ell}{\ell_1, \dots, \ell_m-1, \dots, \ell_p}_q \\
&= \sum_{p=1}^{\ell+1} (-1)^p \sum_{m=1}^p \left[\sum_{\substack{\ell_1+\dots+\ell_{m-1}+\ell_{m+1}+\dots+\ell_p=\ell \\ \ell_i \geq 1}} q^{\ell_1+\dots+\ell_{m-1}} \binom{\ell}{\ell_1, \dots, \ell_{m-1}, \ell_{m+1}, \dots, \ell_p}_q \right. \\
&\quad \left. + \sum_{\substack{\ell_1+\dots+\ell_p=\ell \\ \ell_i \geq 1}} q^{\ell_1+\dots+\ell_{m-1}} \binom{\ell}{\ell_1, \dots, \ell_p}_q \right] \\
&= \sum_{p=1}^{\ell+1} (-1)^p \sum_{m=1}^p \left[\sum_{\substack{\ell_1+\dots+\ell_{p-1}=\ell \\ \ell_i \geq 1}} q^{\ell_1+\dots+\ell_{m-1}} \binom{\ell}{\ell_1, \dots, \ell_{p-1}}_q + \sum_{\substack{\ell_1+\dots+\ell_p=\ell \\ \ell_i \geq 1}} q^{\ell_1+\dots+\ell_{m-1}} \binom{\ell}{\ell_1, \dots, \ell_p}_q \right] \\
&= - \sum_{p=1}^{\ell} (-1)^p \sum_{m=1}^{p+1} \sum_{\substack{\ell_1+\dots+\ell_p=\ell \\ \ell_i \geq 1}} q^{\ell_1+\dots+\ell_{m-1}} \binom{\ell}{\ell_1, \dots, \ell_p}_q \\
&\quad + \sum_{p=1}^{\ell} (-1)^p \sum_{m=1}^p \sum_{\substack{\ell_1+\dots+\ell_p=\ell \\ \ell_i \geq 1}} q^{\ell_1+\dots+\ell_{m-1}} \binom{\ell}{\ell_1, \dots, \ell_p}_q \\
&= - \sum_{p=1}^{\ell} (-1)^p \sum_{\substack{\ell_1+\dots+\ell_p=\ell \\ \ell_i \geq 1}} q^{\ell_1+\dots+\ell_p} \binom{\ell}{\ell_1, \dots, \ell_p}_q \\
&= -q^{\ell} (-1)^{\ell} q^{\frac{\ell(\ell-1)}{2}} = (-1)^{\ell+1} q^{\frac{\ell(\ell+1)}{2}},
\end{aligned}$$

as desired. \square

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